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NOTE ON THE DISCRIMINANT MATRIX OF AN ALGEBRA*

BY L. E. BUSH

The purpose of this note is to extend MacDuffee's normal basis† to a general linear associative algebra.

Let $\mathfrak{A}$ be a linear associative algebra over an infinite field $\mathfrak{F}$, with the basis $e_1, e_2, \cdots, e_n$, and let the constants of multiplication be denoted by c_{ijk}. Let $T_1=(\tau_{rs})$ be the first discriminant matrix of $\mathfrak{A}$, and let $d_h=\sum_k c_{hkk}$. Then $\tau_{rs}=\tau_{sr}=\sum_h c_{srh}d_h$.

If $\mathfrak{A}$ is nilpotent, $d_i=0$, $(i=1, 2, \cdots, n)$,‡ and $T_1=0$. We now suppose that $\mathfrak{A}$ is non-nilpotent and therefore possesses a principal idempotent element e_1.§ Let $\mathfrak{N}$ be the radical of $\mathfrak{A}$, and $\mathfrak{B}$ be the set of elements x of $\mathfrak{A}$ for which $e_1x=0$. Then $\mathfrak{B}<\mathfrak{N}$.¶ It is easily shown that $\mathfrak{A}=e_1\mathfrak{A}+\mathfrak{B}$, where $e_1\mathfrak{A}$ and $\mathfrak{B}$ are algebras whose intersection is zero. Let $e_1\mathfrak{A}=\mathfrak{L}+\overline{\mathfrak{N}}$, where $\overline{\mathfrak{N}}$ is the radical of $e_1\mathfrak{A}$ and $\mathfrak{L}$ is a linear system supplementary to $\overline{\mathfrak{N}}$ in $e_1\mathfrak{A}$. It is not difficult to show that $\mathfrak{N}=\overline{\mathfrak{N}}+\mathfrak{B}$.‖ We may therefore select the basis of $\mathfrak{A}$ as $e_1, e_2, \cdots, e_n$, so that e_1 is the principal idempotent selected above, $e_1, e_2, \cdots, e_\sigma$ is a basis for $\mathfrak{L}$, $e_{\sigma+1}, e_{\sigma+2}, \cdots, e_\rho$ a basis for $\overline{\mathfrak{N}}$, and $e_{\rho+1}, e_{\rho+2}, \cdots, e_n$ a basis for $\mathfrak{B}$. Then $d_i=0, (i>\sigma)$,** and $d_1=\sum_k c_{1kk}=\rho>0$, since if x is in $e_1\mathfrak{A}$, we have $e_1x=x$.

Direct computation shows that if $e_1, e_2, \cdots, e_n$ are subjected to a transformation, $e_i'=\sum_j a_{ij}e_j$, the new d's are given by $d_i'=\sum_j a_{ij}d_j$, $(i=1, 2, \cdots, n)$. Hence if we make the non-singular transformation

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* Presented to the Society, November 28, 1931.

† C. C. MacDuffee, Transactions of this Society, vol. 33, p. 427, proves Theorems 1 and 2 only for algebras with a principal unit. The terminology and notation in this paper are in agreement with that of MacDuffee.

‡ L. E. Dickson, *Algebren und ihre Zahlentheorie*, 1927, p. 108.

§ Dickson, loc. cit., p. 100.

¶ Dickson, loc. cit., p. 100.

‖ This relation follows directly from Dickson, loc. cit., p. 100, Theorem 5, or it can be proved independently.

** Dickson, loc. cit., p. 108.

$$\begin{cases} e_1' = e_1, \\ e_i' = -\dfrac{d_i}{\rho} e_1 + e_i, & (1 < i \leqq \sigma), \\ e_i' = e_i, & (i > \sigma), \end{cases}$$

we obtain $d_1' = \rho$, $d_i' = 0$, $(i > 1)$. This transformation does not alter the bases of $\overline{\mathfrak{N}}$ and $\mathfrak{B}$.

We now have $\tau_{11}' = d_1' = \rho$, $\tau_{r1}' = \tau_{1r}' = c_{1r1}' d_1' = 0$, $(r > 1)$ and, since $\mathfrak{N}$ is an invariant subalgebra of $\mathfrak{A}$, $c_{ijk}' = 0$, (i or $j > \sigma$, $k \leqq \sigma$), and therefore $\tau_{rs}' = \tau_{sr}' = c_{sr1}' d_1 = 0$, ($r$ or $s > \sigma$). This gives

$$T_1' = \begin{Vmatrix} \rho & 0 & 0 & \cdots & 0 & 0 \cdots 0 \\ 0 & \tau_{22}' & \tau_{23}' & \cdots & \tau_{2\sigma}' & 0 \cdots 0 \\ 0 & \tau_{32}' & \tau_{33}' & \cdots & \tau_{3\sigma}' & 0 \cdots 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & \tau_{\sigma 2}' & \tau_{\sigma 3}' & \cdots & \tau_{\sigma\sigma}' & 0 \cdots 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \cdots 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 \cdots 0 \end{Vmatrix}.$$

It is obvious that T_1' can now be reduced to a diagonal matrix by transformations in $\mathfrak{F}$ which leave e_1', $e_{\sigma+1}'$, $e_{\sigma+2}'$, $\cdots$, e_n' invariant, and leave $d_2' = d_3' = \cdots = d_n' = 0$.

We may now reduce the basis of $\mathfrak{N}$ (or if $\mathfrak{A}$ is nilpotent, the basis of $\mathfrak{A}$ itself) to normal form* by a transformation in $\mathfrak{F}$ of the type

$$\begin{cases} e_i' = e_i, & (i \leqq \sigma), \\ e_i' = \sum_{j=\sigma+1}^{n} a_{ij} e_j, & (i > \sigma). \end{cases}$$

Such a transformation does not alter d_i, $(i = 1, 2, \cdots, n)$, e_1, or T_1.

Since the rank of T_1 is σ,† we have proved the following result.

THEOREM. *A basis can be so chosen for* $\mathfrak{A}$ *that*

* Dickson, loc. cit., p. 111.

† Dickson, loc. cit., p. 110.

$$T_1 = \begin{Vmatrix} g_1 & 0 & 0 \cdots 0 & 0 \cdots 0 \\ 0 & g_2 & 0 \cdots 0 & 0 \cdots 0 \\ 0 & 0 & g_3 \cdots 0 & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots g_\sigma & 0 \cdots 0 \\ 0 & 0 & 0 \cdots 0 & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots 0 & 0 \cdots 0 \end{Vmatrix},$$

where the g's are in $\mathfrak{F}$ *and* $d_2 = d_3 = \cdots = d_n = 0$. *If* $\mathfrak{A}$ *is nilpotent, the basis is normal. If* $\mathfrak{A}$ *is not nilpotent,* $d_1 = g_1 \neq 0$, $g_i \neq 0$, $(i = 2, 3, \cdots, \sigma)$, *where* $n - \sigma$ *is the order of the radical of* $\mathfrak{A}$, *and* $e_{\sigma+1}, e_{\sigma+2}, \cdots, e_n$ *is a normal basis for this radical, and* e_1 *is a principal idempotent of* $\mathfrak{A}$.

We may now define a basis of the type whose existence is shown in the above theorem as a *normal basis for* $\mathfrak{A}$. In case $\mathfrak{A}$ is nilpotent, this basis is the ordinary normal basis for a nilpotent algebra; in case $\mathfrak{A}$ has a principal unit, it is MacDuffee's normal basis.

It is evident that a transformation of the form

$$\begin{cases} e_1' = e_1, \\ e_i' = e_i + \sum_{j=\sigma+1}^{n} a_{ij} e_j, & (1 < i \leqq \sigma), \\ e_i' = e_i, & (i > \sigma), \end{cases}$$

leaves unaltered all the properties of the normal basis. But by such a transformation we can make $e_1, e_2, \cdots, e_\sigma$ the basis of a semi-simple subalgebra of $\mathfrak{A}$ having the principal unit e_1.*

COROLLARY. *The normal basis for a non-nilpotent algebra* $\mathfrak{A}$ *can be so chosen that* $(e_1, e_2, \cdots, e_\sigma)$ *is a semi-simple subalgebra of* $\mathfrak{A}$ *having the principal unit* e_1, *and* $(e_{\sigma+1}, e_{\sigma+2}, \cdots, e_n)$ *is the radical of* $\mathfrak{A}$.

As a consequence of the above theorem we can now omit from MacDuffee's Theorem 2 the restriction "*with a principal unit*".

THE OHIO STATE UNIVERSITY

* Dickson, loc. cit., p. 136.

APOLARITY IN THE GALOIS FIELDS OF ORDER 2^n*

BY A. D. CAMPBELL

Let us consider an m-ary quadratic in the Galois fields of order 2^n

$$f(x_1, x_2, \cdots, x_m) \equiv \sum a_{ij}x_ix_j = 0, \tag{1}$$

where

$$i, j = 1, 2, \cdots, m; j \geqq i; a_{ji} = 0 \text{ if } j \neq i.$$

If m is even, the discriminant of (1) is†

$$\Delta \equiv \begin{vmatrix} 0 & a_{12} & a_{13} \cdots a_{1m} \\ a_{12} & 0 & a_{23} \cdots a_{2m} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{1m} & a_{2m} & a_{3m} \cdots 0 \end{vmatrix}. \tag{2}$$

If m is odd, the discriminant of (1) is*†

$$\Delta \equiv \tfrac{1}{2} \begin{vmatrix} 2a_{11} & a_{12} \cdots a_{1m} \\ a_{12} & 2a_{22} \cdots a_{2m} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{1m} & a_{2m} \cdots 2a_{mm} \end{vmatrix}. \tag{3}$$

We note that in the expansion of (2) we shall have terms like $2a_{12}a_{23}a_{34} \cdots a_{1m} \equiv 0$ modulo 2. Hence (2), when expanded, is of the form

$$a_{12}^2a_{34}^2a_{56}^2 \cdots a_{m-1m}^2 + a_{13}^2a_{24}^2 \cdots + \cdots + (a_{12}a_{34}a_{56} \cdots a_{m-1m} + a_{13}a_{24} \cdots + \cdots)^2.$$

Let us consider a pencil of m-ary quadratics

$$\sum(\lambda b_{ij} + \mu a_{ij})x_ix_j = 0, \tag{4}$$

with b_{ij} and a_{ij} like a_{ij} in (1).

* Presented to the Society, December 28, 1931.

† See A. D. Campbell, *The discriminant of the m-ary quadratic in the Galois fields of order* 2^n, Annals of Mathematics, (2), vol. 29 (1928), No. 3, pp. 395–398.

If m is even and we apply (2) to (4), we have

$$(5)\quad \{(\lambda b_{12}+\mu a_{12})(\lambda b_{34}+\mu a_{34})\cdots(\lambda b_{m-1m}+\mu a_{m-1m})+\cdots\}^2.$$

If we equate to zero the square root of the coefficient of $\lambda^2\mu^{m-2}$ in (5), we obtain the invariant

$$(6)\quad b_{12}(a_{34}a_{56}\cdots a_{m-1m}+\cdots)+b_{13}(a_{24}a_{56}\cdots+\cdots)+\cdots=0.$$

If m is odd and we apply (3) to (4), we have

$$(7)\quad \tfrac{1}{2}\begin{vmatrix} 2(\lambda b_{11}+\mu a_{11}) & \lambda b_{12}+\mu a_{12} & \cdots & \lambda b_{1m}+\mu a_{1m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \lambda b_{1m}+\mu a_{1m} & \lambda b_{2m}+\mu a_{2m} & \cdots & 2(\lambda b_{mm}+\mu a_{mm}) \end{vmatrix}.$$

If we equate to zero the coefficient of $\lambda\mu^{m-1}$ in (7), we obtain the invariant

$$(8)\qquad \sum b_{ij}A_{ij}=0,$$

where

$$i,j=1,2,\cdots,m;\ j\geqq i;\ b_{ji}=0 \text{ if } j\neq i,$$

and where A_{ij} is the cofactor of a_{ij} in a determinant like (2), only with m odd. Thus we have

$$A_{11}=\begin{vmatrix} 0 & a_{23} & \cdots & a_{2m} \\ a_{23} & 0 & \cdots & a_{3m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{2m} & a_{3m} & \cdots & 0 \end{vmatrix},\qquad A_{12}=\begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1m} \\ a_{23} & 0 & \cdots & a_{3m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{2m} & a_{3m} & \cdots & 0 \end{vmatrix},\ \text{etc.}$$

We define the polar (or tangent) hyperplane of any point $P'(x_1', x_2', \cdots, x_m')$ with respect to (1) by the equation

$$(9)\qquad \sum\frac{\partial f}{\partial x_i'}x_i=0,\qquad (i=1,2,\cdots,m).$$

To find the equation of (1) in hyperplane coordinates we seek the condition that the tangent hyperplane (9) shall be the same as

$$(10)\qquad \sum u_ix_i=0,$$

and that (10) shall pass through P'. We get equations of the form

$$(11)\qquad \begin{aligned} -\rho u_1 + 2a_{11}x_1' + a_{12}x_2' + \cdots + a_{1m}x_m' &= 0, \cdots, \\ \sum u_i x_i' &= 0. \end{aligned}$$

If m is even, the determinant of the coefficients of the equations (11), considered as equations in the unknowns ρ, x_1', x_2', $\cdots$, x_m', vanishes identically because this determinant is then a skew-symmetric determinant of odd order (modulo 2). The vanishing of this determinant means that, for m even, the hyperplane (9) always passes through P' even when (9) is only a polar (and not a tangent) hyperplane with respect to (1). Therefore, for m even, we define the equation of (1) in hyperplane coordinates as having the form

$$(12)\qquad \tfrac{1}{2}\begin{vmatrix} 2a_{11} & a_{12} & \cdots & a_{1m} & u_1 \\ a_{12} & 2a_{22} & \cdots & a_{2m} & u_2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_{1m} & a_{2m} & \cdots & 2a_{mm} & u_m \\ u_1 & u_2 & \cdots & u_m & 0 \end{vmatrix} \equiv \sum A_{ij}' u_i u_j = 0,$$

where $A_{ij}'(i \neq j)$ is defined as A_{ij} for (8) and $A_{ji}' = 0$ if $j \neq i$, but $A_{ii}' = \frac{1}{2}A_{ii}$.

For m odd, we define the equation of (1) in hyperplane coordinates as having the form

$$(13)\qquad \begin{vmatrix} 0 & a_{12} & \cdots & a_{1m} & u_1 \\ a_{12} & 0 & \cdots & a_{2m} & u_2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_{1m} & a_{2m} & \cdots & 0 & u_m \\ u_1 & u_2 & \cdots & u_m & 0 \end{vmatrix} \equiv \sum A_{ii} u_i^2 = 0,$$

where A_{ii} is defined as for (8). We note that, for m even, there is no term in (5) of the form $\alpha\lambda\mu^{m-1}$. Even if we define apolarity as the relation given by the invariant (6), this has no simple geometrical meaning.

For m odd, the equations

$$(14)\qquad \begin{aligned} 2a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= 0, \\ a_{12}x_1 + 2a_{22}x_2 + a_{23}x_3 + \cdots + a_{2m}x_m &= 0, \cdots, \text{(modulo 2)} \end{aligned}$$

have a common solution, since the determinant of the coeffi-

cients vanishes (being skew-symmetric and of odd order). Geometrically, this means that all the polar and tangent hyperplanes of (1) pass through a common point P, for m odd.

If $P(X_1, X_2, \cdots, X_m)$ is this common solution, we have

$$X_1 = k_1A_{11},\ X_2 = k_1A_{12}, \cdots, X_m = k_1A_{1m},$$
$$X_1 = k_2A_{12},\ X_2 = k_2A_{22}, \cdots, X_m = k_2A_{2m},$$
$$X_1 = k_3A_{13}, \text{ etc.}$$

But A_{11} has the form α_{11}^2, being a skew-symmetric determinant of even order, like (2). Similarly, $A_{22} = \alpha_{22}^2, \cdots, A_{mm} = \alpha_{mm}^2$. Also we have

$$X_1X_2 = k_1k_2A_{12}^2, X_1 = \frac{k_1A_{12}^2}{A_{22}} = k_1A_{11};$$

hence $A_{12}^2 = A_{11}A_{22} = \alpha_{11}^2\alpha_{22}^2$, so that $A_{12} = \alpha_{11}\alpha_{22}$. But

$$X_1 = k_2A_{12} = k_2\alpha_{11}\alpha_{22} = k_1A_{11} = k_1\alpha_{11}^2;$$

therefore

$$k_1\alpha_{11}^2 = k_2\alpha_{11}\alpha_{22}, \text{ and } \frac{k_1}{k_2} = \frac{1/\alpha_{11}}{1/\alpha_{22}},$$

so that $k_1 = c/\alpha_{11}$, and $k_2 = c/\alpha_{22}$, where c is an arbitrary constant. Finally we have

$$X_1 = k_1A_{11} = \frac{c}{\alpha_{11}}\alpha_{11}^2 = c\alpha_{11},$$

and $X_2 = c\alpha_{22}$. Similarly

$$A_{1i}^2 = \alpha_{11}^2\alpha_{ii}^2, \qquad \frac{k_1}{k_i} = \frac{1/\alpha_{11}}{1/\alpha_{ii}},$$

so that $k_1 = c/\alpha_{11}$ and $k_i = c/\alpha_{ii}$; therefore $X_i = c\alpha_{ii}$.

From the above discussion we see that the equations (14) have the common solution

$$(15)\quad P(X_1, X_2, \cdots, X_m) = P\{(A_{11})^{1/2}, (A_{22})^{1/2}, \cdots, (A_{mm})^{1/2}\},$$

with A_{ii} defined as for (8). If we call (8) the relation of apolarity between the point quadratic $\sum b_{ij}x_ix_j = 0$ and the quadratic in

hyperplane coordinates given by (13), we see that (8) is the condition that $P(X_i \equiv (A_{ii})^{1/2})$ shall lie on the apolar quadratic $\sum b_{ij}x_ix_j = 0$.

Finally we note that if we expand (3) m times, first using the elements of the first column and their cofactors, then the elements of the second column and their cofactors, and in like manner to the last column, and if we then add our results and equate Δ to zero (removing the odd factor m), we get (3) in the form $\sum a_{ij}A_{ij} = 0$, where a_{ij} and A_{ij} are the same as for (8). This shows us that for (1) to be a degenerate quadratic, when m is odd, the point P in (15) must lie on (1). There is no similar simple geometrical description when m is even and (1) is degenerate.

SYRACUSE UNIVERSITY

A CLASS OF UNIVERSAL FUNCTIONS*

BY GORDON PALL

Let a, b, c, d be integers, $a \neq 0$. The function $f(x, y)$ defined by the equation

$$f(x, y) = axy + bx + cy + d \tag{1}$$

will be called *universal* if $f(x, y)$ represents all integers for integral values of x and y.

THEOREM 1. *A necessary and sufficient condition for* (1) *to be universal is that*

$$b \equiv \pm 1 \text{ or } c \equiv \pm 1 \pmod{a}, \tag{2}$$

or $a = 6$, $b \equiv \pm 3$, $c \equiv \pm 2 \pmod 6$, *or vice versa for* b *and* c.†

The sufficiency is evident. For, if $b = \pm 1 + Ba$,

$$f(x, -B) = \pm x + d - Bc.$$

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* Presented to the Society, December 28, 1931.

† The writer was led to the exceptional form $6xy+3x+2y$ as in the analysis below, but through an oversight he thought it did not represent 7. The error was, fortunately, pointed out by W. L. G. Williams before this paper went to press.

By replacing x by $y-h$ and y by $y-k$, we can vary the coefficients b and c modulo a. By altering the sign of x or y or both we get a, b, c of like sign. By changing the sign of f we get $a>0$. Finally the constant d may be dropped without affecting the generality. Hence f is reduced to a form in which

$$0 \leqq b \leqq \tfrac{1}{2}a, \quad 0 \leqq c \leqq \tfrac{1}{2}a, \quad d = 0. \tag{3}$$

The process of reduction does not change the absolute value of either b or c (mod a). Hence the theorem will follow if proved for forms satisfying (3).

The theorem is evident for (3) if $a=1, 2, 3, 4$ or if either b or c is equal to 1 or 0. Hence we may suppose $a \geqq 5$, $b \geqq c \geqq 2$. If $|x| \geqq 2$, $|y| \geqq 2$, then

$$|axy + bx + cy| \geqq 2a$$

by (3). If x or y has the values $-1, 0, 1$ the only values of f which may equal 1 or 2 are $a-b-c, b, c$. Hence $a-b-c=1$, and $c=2$, whence either

$$a = 5, \ b = c = 2; \text{ or } a = 6, \ b = 3, \ c = 2.$$

In the first case f fails to represent 3. In the second case f is evidently universal, since

$$6xy+3x+2y+1=(3x+1)(2y+1),$$

and $3x+1$ represents either 2^h or -2^h.

We consider an extension to functions (1) which are $\geqq 0$ for all integers x, $y \geqq 0$. Evidently these occur if and only if $a>0$, $b \geqq 0$, $c \geqq 0$, $d \geqq 0$. Then $f(x, y)$ represents 0 for integers $\geqq 0$ if and only if $d=0$, and then represents 1 only if b or c is 1. Writing

$$\alpha=a, \ \beta=b, \ \xi=x+1, \ \eta=y+1,$$

we have the following result.

THEOREM 2. *Let α, β denote integers, $\alpha>0, \beta \geqq 0$. The only functions of the type* (1) *which represent only positive integers for positive integers ξ and η, and represent all such integers for such ξ and η, are*

$$\phi(\xi, \eta) = \xi\eta + (\alpha - 1)(\xi - 1)(\eta - 1) + (\beta - 1)(\xi - 1), \tag{4}$$

and $\phi(\eta, \xi)$.

The function obtained from (4) with $\beta=1$, namely

$$\psi(x, y) \equiv xy + (\alpha - 1)(x - 1)(y - 1), \tag{5}$$

is of special interest in view of the property

$$\psi(\psi(x, y), z) = \psi(x, \psi(y, z)). \tag{6}$$

If we call $\psi(x, y)$ the α-product of x and y and denote it alternatively by $x \circ y$, then

$$x \circ y = y \circ x, \quad (x \circ y) \circ z = x \circ (y \circ z),$$

and the positive integers may be studied under α-multiplication. We have $1 \circ y = y$, so that 1 acts as identity element. We call the positive integer $y > 1$ an α-prime if its only divisors under α-multiplication are 1 and y. For example, 2, 3, 4, 6, 7, 9, 10, 12, 15, 16, 19, $\cdots$ are 2-primes; but 5, 8, 11, 13, 14, 17, 18, $\cdots$ are 2-composite since, if $\alpha = 2$, $5 = 2 \circ 2$, $8 = 2 \circ 3$, $\cdots$, $13 = 3 \circ 3$, $\cdots$. For any α there are infinitely many α-primes.

It is easy to see that α-decomposition (apart from order) into α-primes is unique if and only if $\alpha = 1, 2$. That it is unique if $\alpha = 2$ is plain from the equivalence of the equations

$$n = 2xy - x - y + 1, \quad 2n - 1 = (2x - 1)(2y - 1).$$

It follows also from the equivalence that p is a 2-prime if and only if $2p - 1$ is an ordinary prime.

If $\alpha = 3$, however, $4 \circ 8 = 2 \circ 19$ but 2, 4, 8 and 19 are distinct 3-primes. It is easy to construct the ideal divisors, restoring unique factorization. Generally, the theory is equivalent to that of the set of numbers $\alpha x + 1$, $x = 0, 1, 2, \cdots$, under ordinary multiplication.

McGill University

THE PLANE FIGURE OF SEVEN REAL LINES*

BY H. S. WHITE

1. *Introduction.* A set of real lines, finite in number, in the plane of projective geometry, divides the plane into convex polygons, each bounded by segments of those lines. If no three lines meet in a point, then three lines form 4 polygons (triangles); four lines, 7 polygons; n lines, $(n^2-n+2)/2$ polygons. For any given diagram of this sort, the triangles, quadrilaterals, etc., may be counted; then if the lines are allowed to move freely in the plane, every polygon will retain the same number of sides until three or more points of intersection (or two pairs) come to coincide. Exclude this situation, and we have as invariants the integers showing the numbers of polygons of 3, 4, 5, $\cdots$, n sides; also a scheme showing contiguities. We shall exhibit such schemes for a set of 7 lines; and inquire how many kinds of (non-equivalent) sets exist, when a one-to-one relation between lines and polygons of two sets constitutes equivalence.

2. *Unique Sets,* $n=3$, 4, 5. Since central projection is a particular kind in the group of transformations that we here admit, and since four lines in a plane are projective to any other set of four—barring cases where three lines are copunctual—the arrangement of any such set is typical of all. Three lines form four triangles; three of them have infinitely long boundaries, but that is projectively of no account; and each is adjacent to all the others. A fourth line intersects three segments exterior to one of these triangles, say T_1, and divides each of the other triangles into a quadrilateral adjacent to T_1 and a triangle having no sides, but only one vertex, in common with T_1. Otherwise stated (Fig. 1), *four lines in a plane, no three in any point, constitute twelve segments bounding four triangles and three convex quadrilaterals. Each quadrilateral is adjacent to all four triangles, but two triangles have in common only one vertex, while two quadrilaterals have in common two opposite vertices.*

Five lines are not necessarily *projective* to an arbitrarily se-

* Presented to the Society, October 31, 1931.

lected set of five, but they are *equivalent* under the group here considered. For a fifth line intersects the four above mentioned upon four segments, each of which separates a triangle from a quadrilateral. Hence the fifth line crosses two triangles and two quadrilaterals; say in cyclic order $T_1Q_1T_2Q_2$. The two sides of T_1 and the two of T_2 which it meets contain all 5 termini (ver-

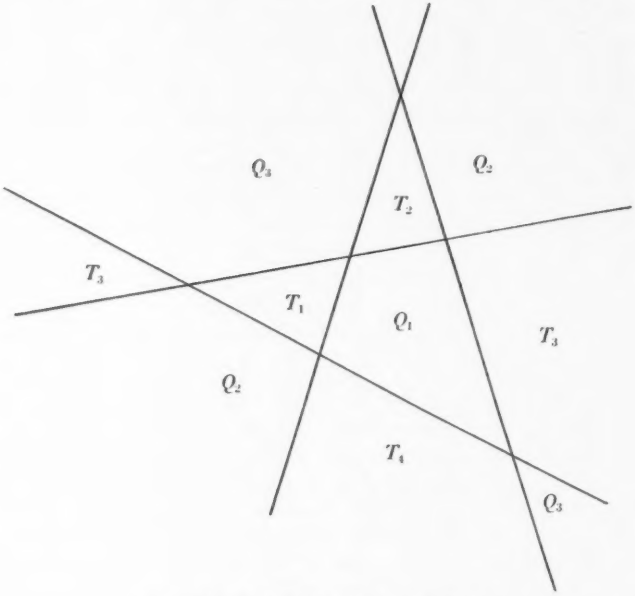

FIG. 1. Lines 4, Quadrilaterals 3, Triangles 4.

tices); hence two sides, one in each triangle, contain the common vertex. But those two must be *adjacent* sides of either Q_1 or Q_2, say of Q_1; and the other two segments cut by the fifth line have no terminus in common, and are therefore *opposite* sides of Q_2. Accordingly Q_2 is divided into two quadrilaterals, and Q_1 into a triangle and a pentagon. The triangles T_1 and T_2 are divided, each into a triangle and a quadrilateral.

Among the 11 *convex polygons whose boundaries, but no interior*

points, lie in five straight lines of a plane with 10 *distinct intersections, there are*

1 *pentagon*;
5 *triangles, each adjacent to the pentagon*; *and*
5 *quadrilaterals, each adjacent to two triangles and adjacent to two other quadrilaterals.*

This description is most easily visualized by the aid of a regular pentagon with sides produced indefinitely. In such a figure all 10 intersections are in the finite plane.

For convenience of allusion, the 20 segments of the 5 lines may be assigned to 3 classes, viz.:

5 primary, separating the pentagon from triangles,
10 secondary, separating the triangles from quadrilaterals,
5 tertiary, separating each, two quadrilaterals.

Note also that each of the 3 classes forms a single continuous broken line, two of five parts, both of even character, one of ten, of odd character. Each quadrilateral has two adjacent sides secondary segments; and two tertiary, adjacent.

3. *More than Five Lines.* By selecting any five lines of a set as the initial five, we have a unique description for their relative situation. Adding a sixth line, we shall find four different arrangements possible. When there are seven in a set, there are five to be selected at random, and afterward reasons may be found for discriminating the sixth from the seventh. From the diagram of any five lines we may safely write down two observations:

I. As no two triangles have a side in common, no sixth line can cross three triangles. It can cross either two or one.

II. There are lines that meet only tertiary segments, crossing no triangle.

Hence a sixth line will fall into one of the following 4 species (see Fig. 2).

(a) A line cutting 2 primary segments, 2 secondary, 1 tertiary. This species will be divided according as the primary segments are consecutive or not, on the pentagon boundary.

(b) A line cutting 4 secondary segments and 1 tertiary. The two triangles must have one vertex in common.

(c) A line cutting 2 secondary and 3 tertiary segments. This crosses one triangle and four quadrilaterals.

(d) A line cutting 5 tertiary segments. (See II above.)

ad(a) In Fig. 2, line *a* is shown cutting consecutive sides of the pentagon, thus forming one hexagon.

ad(b) Choosing line *x* for the sixth or secant line, and *pqrsb* for the pentagon, we have a secant cutting non-consecutive sides of the pentagon, a case under (a).

ad(c) Exchange roles of *x* and *c*. Again case (a).

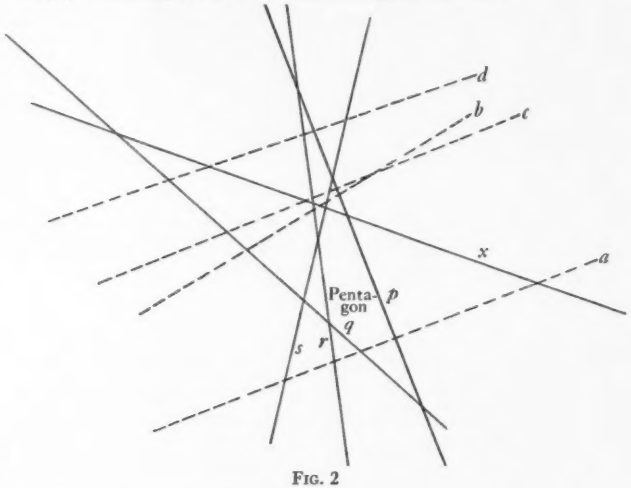

FIG. 2

ad(d) The sixth line cuts adjacent sides in each of the 5 quadrilaterals, forming 5 new pentagons. Every segment is now a side of some one pentagon. Therefore any *seventh* line will necessarily cross 3 pentagons, and a new choice of the initial five lines and the sixth will reduce this to case (a).

Incidentally we may tabulate the sets of polygons formed by five or six lines, convex polygons having no points of the lines in their interior.

Polygons of		6	5	4	3
Five lines		0	1	5	5
Six lines	a	1	0	9	6
	b	0	2	8	6
	c	0	3	6	7
	d	0	6	0	10

sides formed by as in Figure 2.

Division of the subject. When, in a set of seven lines, any six form a hexagon, the discussion may utilize that hexagon as unique initial figure. This part of the subject has been investigated by Miss L. D. Cummings. I shall survey only sets of seven lines in which no six form a convex hexagon.

4. *Notation and Method.* Let there be given a pentagon and a secant crossing two segments of its boundary, not coterminous (consecutive), since then a hexagon would be formed. This secant must cross one and only one tertiary segment at a point which we shall call A, and two secondary segments belonging to two triangles. Of those triangles, one or both must have seg-

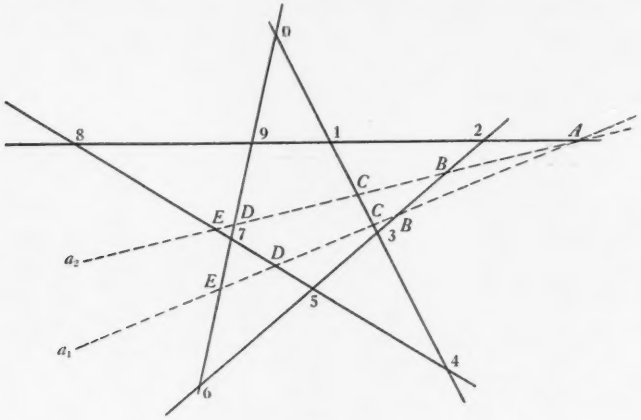

FIG. 3

ments in their boundary collinear with that unique tertiary segment, since otherwise their four secondary boundaries would lie in the other four lines of the pentagon and the two triangles would have one vertex in common, whence the primary sides would be consecutive. Call the secant line a_1 when it crosses only one triangle with a side collinear to A; call it a_2 when it crosses both triangles having secondary sides collinear with A. (See Fig. 3.)

The Secant is a_1. Designate by 12 the secondary side, collinear with A, of the first triangle crossed by the secant; by 23 its other

secondary side, crossing the secant at B. Mark the other intersections of the secant so that the five are in order on that secant, $ABCDE$. Two vertices of the pentagon are already marked; let the five vertices in order on the boundary of the pentagon be 1, 3, 5, 7, 9. On the broken line of secondary segments insert 4, 6, 8, 0 in due order between odd digits. Now arrange in continuous broken-line divisions the segments that cross the secant, beginning with any point terminating only one such segment. Thus the secant a_1 will be recorded by 8231.576.

The Secant is a_2. Proceed in the same way. The choice of point 1 is in this case ambiguous, and there are two admissible sets of marks. If the one set is

$$1234567890ABCDE,$$

then the other set is

$$9876543210AEDCB.$$

We may call this the symmetric case, the other unsymmetric. For a_2, the recorded position will be 978231.

It remains to form all possible records for a seventh line which shall not cut consecutive sides of this pentagon, using both numerals and letters. Every set of seven lines can be separated in 21 ways into five-and-two; then whenever both lines of the *two* cross the pentagon of the others, two records will be permitted by exchange of sixth and seventh line roles. Conceivably, therefore, 42 records might denote a single set of 7 lines. Always there will be at least one record, since at least one choice of five lines from the seven will leave a secant a_1 or a_2.

5. *Experimental or Observational Geometry.* Since a definite order is essential to an exhaustive census, I have taken small sets of positions for the seventh line; five for case a_1, three for case a_2 (symmetric), collecting together those that cut the same segment of the sixth line. Each position is derived from one already obtained (starting from coincidence with the sixth secant) by sweeping over one of the ten vertices of the complete pentagon. This is not complicated, and after the list is filled out for each segment, AB, BC, $\cdots$, then I erase all that show two consecutive segments from the boundary of either pentagon—the fundamental one; or, for a_1, that second one with vertices $AB348$; or the third, $CD791$. In this way reduced lists are

fixed. For example, in case a_1, where the sixth secant designated by 8231.576, if the seventh crosses the segment AB, it may be any one of the following:

$2AB.3456E$	$62AB.109.84$	$6048.AB21$
$BA260.345$	$4897.C1.AB2$	$60.345.AB21$
$84062AB$	$48.AB2109$	$840.56E.2AB.$

For each of these in turn 21 duplicate diagrams are made, beginning with the first. Each diagram has a pentagon and two secants outlined in colored crayon. When hexagons appear, six or ten to any set, or in one case the heptagon, those are laid aside; as are also the sketches in which both secants are external to the pentagon (cases (c) or (d)). For each sketch not eliminated, either one or two records are determined on the basis of §4, and these are checked off on the list and classed as equivalents.

By this means, three systems and no more are found in which no hexagon occurs. Representative descriptions of those three are the following:

8231.576 and $62AB.109.84$, 8231.576 and $2AB.3456E$,

8231.576 and $84062AB$.

This work though laborious is not tedious. Mistakes and omissions occur, but the multitude of equivalences makes their discovery almost certain. Aside from the intrinsic interest of such a catalog, we have undertaken the problem for the purpose of studying Aronhold sets of seven double-tangents of a plane quartic curve. As to polygonal division of the projective plane by lines, we note for $n=5$, 6, or 7 the contrast between our results and the much larger number of systems that satisfy Euler's equation. The polygons in these three non-hexagonal sets of seven lines are as follows, in order as above.

No. of sides	7	6	5	4	3
1	0	0	3	12	7
2	0	0	4	10	8
3	0	0	6	6	10

VASSAR COLLEGE

POLYGENIC FUNCTIONS OF HYPERCOMPLEX VARIABLES

BY P. W. KETCHUM AND TED MARTIN

1. *Introduction.* The purpose of this paper is to generalize to hypercomplex variables some of the results obtained by Kasner in the papers recently published by him on polygenic functions of the complex variable z.* Kasner has applied the term *polygenic* to any function of the form

$$f(z) = \phi(x, y) + i\psi(x, y),$$

where ϕ and ψ are arbitrary functions (except for suitable continuity and differentiability assumptions) of the real variables x, y, the Cauchy-Riemann conditions not being assumed.

If the first derivative df/dz, which is denoted by $\gamma = \alpha + i\beta$, be plotted in the α, β plane, the locus of the points γ corresponding to a given point z is a circle

$$\left(\alpha - \frac{\phi_x + \psi_y}{2}\right)^2 + \left(\beta - \frac{-\phi_y + \psi_x}{2}\right)^2 = \left(\frac{\phi_x - \psi_y}{2}\right)^2 + \left(\frac{\phi_y + \psi_x}{2}\right)^2.$$

To the ∞^2 points of the z-plane correspond ∞^2 circles. This congruence of circles is called the *Kasner derivative congruence.* (In special cases where the function f is monogenic the circles all shrink to points.)

Hedrick, Ingold, and Westfall† developed the theory of non-analytic functions of a complex variable in a paper published in 1923 and E. R. Hedrick in a later paper‡ pointed out relations

* E. Kasner, *Theory of polygenic functions*, Science, vol. 66 (1927), pp. 581–582, and Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 75–82; *Note on the derivative circular congruence of a polygenic function*, this Bulletin, vol. 34 (1928), pp. 561–565.

† Hedrick, Ingold and Westfall, *Theory of non-analytic functions of a complex variable*, Journal de Mathématiques, (9), vol. 2 (1923), pp. 327–342.

‡ E. R. Hedrick, *On derivatives of non-analytic functions*, Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 649–654.

existing between the papers of Kasner and the paper of Ingold, Westfall, and himself.

Edna E. Kramer* has developed a theory analogous to this for polygenic functions of the dual variable $w=u+jv$,

$$f(w) = \phi(u, v) + j\psi(u, v).$$

Mapping the derivative df/dw in the plane of $\gamma=\alpha+j\beta$, she obtained the congruence

$$\beta = \frac{\phi_v\psi_u - \phi_u\psi_v}{\phi_v} + \frac{\psi_v + \phi_u}{\phi_v}\alpha - \frac{1}{\phi_v}\alpha^2.$$

This equation represents a congruence of cycles (directed circles) in the coordinate system she used, but in the Cartesian coordinate system the congruence would be one of parabolas with axes parallel to the β-axis.

Analytic functions of hypercomplex variables have been studied by Scheffers† for algebras which are commutative and associative. Ketchum‡ has extended to analytic functions of hypercomplex variables all the important elementary theorems of complex variables. Antonne§ has studied monogenic functions of variables in non-commutative algebras.

In § 2 we shall give a generalization of the Kasner derivative congruence and in § 3 we shall study some of the degeneracies of this generalized congruence.

2. *The Kasner Derivative Congruence.* We will denote the general hypercomplex variable by

$$(1) \qquad w = \sum_{i=1}^{n} x_i e_i; \quad e_i e_j = \sum_{s=1}^{n} \gamma_{ijs} e_s, \quad (i, j = 1, \cdots, n),$$

where the units e_i are linearly independent with respect to the complex domain, the constants γ_{ijs} are real numbers, and the x's are complex variables.

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* Edna E. Kramer, *Polygenic functions of the dual variable* $w=u+jv$, American Journal of Mathematics, vol. 52 (1930), pp. 370–376.

† Leipziger Berichte, vol. 45 (1893), p. 828.

‡ P. W. Ketchum, *Analytic functions of hypercomplex variables*, Transactions of this Society, vol. 30 (1928), pp. 641–667.

§ Journal de Mathématiques, (6), vol. 3 (1907).

Consider the polygenic function

$$f(w) = \sum_{i=1}^{n} e_i f_i(x_1, \cdots, x_n), \tag{2}$$

where the f_i are continuous functions of the variables $x_1, \cdots, x_n$ and possess continuous derivatives of the first order. We have the two derivatives

$$df = f'dw, \quad df = dwf', \tag{3}$$

called the *left-hand* and *right-hand* derivatives respectively. If the algebra is commutative, the two derivatives are equal. We will, however, not assume that the algebra is commutative and we will develop the theory for the left-hand derivative. A parallel development could be carried out for the right-hand derivative, the first two subscripts on the γ's being interchanged throughout. From (3_1) we obtain the relation

$$\sum_{i=1}^{n} e_i df_i = \sum_{k,j=1}^{n} e_k f_k' e_j dx_j. \tag{4}$$

Hence

$$\sum_{i,j=1}^{n} e_i \frac{\partial f_i}{\partial x_j} dx_j = \sum_{i,j,k=1}^{n} f_k' e_i \gamma_{kji} dx_j. \tag{5}$$

Since the e's are linearly independent we can equate coefficients. We obtain n linear homogeneous equations in the n variables dx_j,

$$\sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} - \sum_{k=1}^{n} f_k' \gamma_{kji} \right) dx_j = 0, \qquad (i = 1, \cdots, n), \tag{6}$$

which have a solution other than $0, \cdots, 0$ if and only if the determinant

$$\left| \frac{\partial f_i}{\partial x_j} - \sum_{k=1}^{n} f_k' \gamma_{kji} \right| = 0. \tag{7}$$

If (7) be considered as a function of the n variables f_k', it represents an n-parameter family of hypersurfaces, M_n^n. That is, for every point $x_1, \cdots, x_n$ there corresponds a hypersurface of M_n^n. Equation (7) is a generalization of the Kasner congruence

for functions of ordinary complex variables. We propose to call this the *Kasner derivative congruence* for polygenic functions of hypercomplex variables.

For a given direction of differentiation, that is, for a given value of dw, unique values of f' will be determined by (3_1), if and only if dw is such a number in the algebra that left-hand division by it is possible. In other words f' will be determined uniquely if dw is not a nilfactor. A necessary and sufficient condition that unique left-hand division by dw is possible, as can be seen from (6), is that the determinant

$$\Delta'_{dw} = \left| \sum_{j=1}^{n} \gamma_{kji} dx_j \right| \neq 0. \tag{8}$$

If Δ'_{dw} is zero, we must consider the rank of the two matrices

$$M_1: \left\| \sum_{j=1}^{n} \gamma_{kji} dx_j \right\|; \quad M_2: \left\| \sum_{j=1}^{n} \gamma_{kji} dx_j, \sum_{s=1}^{n} \frac{\partial f_i}{\partial x_s} dx_s \right\|.$$

If the rank r of the matrix M_1 is less than the rank of the augmented matrix certain restrictions are placed on the dx_i, that is, the derivative does not exist for certain paths of approach. If, however, M_1 and M_2 have the same rank r, which may happen at certain points or may be true for certain functions everywhere, then values of $n-r$ of the unknowns f_k' may be assigned at pleasure and the others will be uniquely determined.

We will give an example of a function in an algebra of two units such that restrictions are placed on the direction of differentiation.

Let us consider the variable $w=ix+jy$ and the function $f(w) = i\phi(x, y)+j\psi(x, y)$ in the algebra with the multiplication table

	i	j
i	i	i
j	j	j

The matrix $\|\sum_i \gamma_{jki} dx_j\|$ for this variable is

$$\begin{Vmatrix} dx & dx \\ dy & dy \end{Vmatrix},$$

which is of rank 1. The corresponding augmented matrix for this variable is

$$\left\| \begin{matrix} dx & dx & \phi_x dx + \phi_y dy \\ dy & dy & \psi_x dx + \psi_y dy \end{matrix} \right\|.$$

This matrix is of rank greater than 1 unless the relation

$$\left| \begin{matrix} dx & \phi_x dx + \phi_y dy \\ dy & \psi_x dx + \psi_y dy \end{matrix} \right| = 0$$

holds, that is, unless

$$m^2\phi_y + m(\phi_x - \psi_y) - \psi_x = 0.$$

Thus we see that a restriction is imposed on the direction of differentiation.

This restriction also appears in the actual process of forming the right-hand derivative. We have

$$df = dwf'(w) = (idx + jdy)(i\alpha + j\beta),$$
$$i\phi_x + i\phi_y m + j\psi_x + j\psi_y m = i\alpha + i\beta + j\alpha m + j\beta m.$$

Equating coefficients, we obtain the two equations

$$\phi_x + m\phi_y = \alpha + \beta, \quad \psi_x + m\psi_y = m(\alpha + \beta),$$

which show that the relation

$$m^2\phi_y + m(\phi_x - \psi_y) - \psi_x = 0$$

must hold.

If we take the function $f(w)$ to be one such that $\phi_y \equiv \psi_x \equiv 0$ and $\phi_x \equiv \psi_y$, then the ranks of M_1 and M_2 are always the same, namely one, and the value of one of the two unknowns α or β may be assigned at pleasure and the other will be uniquely determined. If at certain points the relations $\phi_y = \psi_x = 0$, $\phi_x = \psi_y$ hold, then at those points the right-hand derivative will exist for every direction of approach and the value of one of the two unknowns may be assigned as in the previous case.

If we make the substitutions

$$(9) \qquad a_{ij} = \sum_{k=1}^{n} f_k' \gamma_{kji}, \quad b_{ij} = -\frac{\partial f_i}{\partial x_j},$$

we can write the equation (7) as

$$(10)\qquad \begin{aligned} &|a_{ij}| + \sum_{\substack{r=1\\ j\neq r}}^{n} |a_{ij}b_{ir}| + \sum_{\substack{r,s=1\\ j\neq r,s}}^{n} |a_{ij}b_{ir}b_{is}| + \cdots \\ &\quad + \sum_{\substack{r=1\\ j\neq r}}^{n} |a_{ir}b_{ij}| + |b_{ij}| = 0. \end{aligned}$$

A consideration of equation (10) shows that the first term, $|a_{ij}|$, represents the terms of the nth degree in the variables f_k', and since it depends only on the γ's, it is characteristic of the hypervariable. The determinant $|a_{ij}|$ is seen to be $\Delta_{f'}$ where Δ_f' is the right-hand determinant corresponding to (8) for right-hand division. The equation

$$|a_{ij}| = 0$$

determines the general nature of the hypersurfaces, since it determines the intersections of the hypersurfaces with the hyperplane at infinity. Thus we see that for a given algebra the hypersurfaces corresponding to any functions in the algebra are of the same general nature. The constant term, $|b_{ij}|$, involves only the derivatives of the function considered and is independent of the hypervariable. The intermediate terms depend on both the hypervariable and the function.

3. *Degeneracies of the Congruence.*

THEOREM. *The hypersurfaces* (7) *will all go through the origin if and only if the jacobian of* $f_1, \cdots, f_n$ *vanishes.*

From the equation of the congruence written in the form of equation (10) we see that the hypersurfaces all go through the origin when and only when the constant term is zero, that is, in view of the second relation of (9), when and only when the functions $f_1, \cdots, f_n$ are functionally dependent.

If $|b_{ij}| = 0$, then for a fixed point in the plane of $x_1, \cdots, x_n$ the corresponding hypersurface in the plane of $f_1', \cdots, f_n'$ will be tangent to the hyperplane

$$(11)\qquad \sum_{\substack{r=1\\ j\neq r}}^{n} |a_{ir}b_{ij}| = 0$$

at the origin.

If the rank of the jacobian matrix is $n-2$, that is, if there exist two relations (and not more) between the f's which do not involve the x's, the hypersurfaces will all have the origin as a double point, and terms of the second degree of (10) tell us the nature of the singular point.

We note that equation (7) would be satisfied if every element vanished. This would lead to n^2 linear equations in the quantities f_k'. It is known that if the algebra is commutative and associative, these n^2 equations have a unique solution for all analytic functions of w. The derivative congruence would in this case degenerate into a congruence of points, the function being monogenic.

UNIVERSITY OF ILLINOIS

ON FINITE AND INFINITE COMPLETELY MONOTONIC SEQUENCES*

BY I. J. SCHOENBERG

1. *Introduction.* As one of several important results concerning the problem of moments for a finite interval, Hausdorff† has proved the following theorem.

The problem of moments, which is to find a real function $\chi(t)$ for $0 \leqq t \leqq 1$, such that

$$\mu_n = \int_0^1 t^n d\chi(t), \qquad (n = 0, 1, 2, \cdots), \tag{1}$$

in which the sequence $\mu_0, \mu_1, \mu_2, \cdots$ is given in advance, has a monotonic non-decreasing solution, if and only if

$$\Delta^l \mu_m = \sum_{\nu=0}^{l} (-1)^\nu \binom{l}{\nu} \mu_{m+\nu} \geqq 0, \qquad \text{for } l, m = 0, 1, 2, \cdots, \tag{2}$$

in which case the sequence is said to be completely monotonic.

The same problem (1) *has a solution $\chi(t)$ which is of bounded variation, if and only if*

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* Presented to the Society, November 28, 1931.

† *Ueber das Momentenproblem für ein endliches Interval*, Mathematische Zeitschrift, vol. 16 (1923), pp. 220–248. We refer to §§1 and 2.

$$(3) \qquad \sum_{m=0}^{p} \binom{p}{m} \left| \Delta^{p-m}\mu_m \right| = O(1) \text{ for } p \to \infty .^*$$

Hausdorff proves this theorem starting from the system (1), which defines a positive linear functional operation by interpolation in function space, and proves that the conditions (2) are necessary and sufficient for the existence of such a functional operation defined by the function $\chi(t)$. In this note we shall proceed in the opposite way. We shall solve the set (2) of linear inequalities in infinitely many variables and shall find that its most general solution may be represented parametrically in the form (1). In another paper the same point of view will be applied to similar but more general problems considered by Carathéodory, F. Riesz, and Hausdorff.

2. *Finite Sequences.* It seems natural to start first with the following problem. Let us call the finite sequence $\mu_0, \mu_1, \cdots, \mu_p$ *completely monotonic if*

$$(4) \quad \Delta^l\mu_m \geqq 0 \text{ for } l, m = 0, 1, 2, \cdots, p, \text{ with } l + m \leqq p .\dagger$$

We seek a parametric representation for the most general solution of the set (4) *of linear inequalities.*

Such a representation is readily found. The identity $\Delta^l\mu_m = \Delta^l\mu_{m+1} + \Delta^{l+1}\mu_m$ shows that the set (4) is a consequence of its sub-set

$$(5) \qquad \Delta^{p-m}\mu_m \geqq 0, \quad \text{for } m = 0, 1, 2, \cdots, p,$$

which means that the $p+1$ linear forms

$$(6) \quad \begin{aligned}
\rho_{p,0} &= \mu_0 - \binom{p}{1}\mu_1 + \binom{p}{2}\mu_2 - \cdots + (-1)^p \mu_p, \\
\rho_{p,1} &= \mu_1 - \binom{p-1}{1}\mu_2 + \cdots + (-1)^{p-1}\mu_p, \\
\rho_{p,2} &= \mu_2 - \cdots + (-1)^{p-2}\mu_p, \\
&\vdots \\
\rho_{p,p-1} &= \mu_{p-1} - \mu_p, \\
\rho_{p,p} &= \mu_p,
\end{aligned}$$

* This statement is certainly verified for the previous case, because (2) gives $\sum_{m=0}^{p}\binom{p}{m}\left|\Delta^{p-m}\mu_m\right| = \sum_{m=0}^{p}\binom{p}{m}\Delta^{p-m}\mu_m = \mu_0$.

† This definition is justified by the fact that these are all the higher differences which may be derived from our finite sequence.

should be non-negative. The linear transformation inverse to (6) is

$$
\begin{aligned}
\mu_0 &= \rho_{p,0} + \binom{p}{1}\rho_{p,1} + \binom{p}{2}\rho_{p,2} + \cdots + \rho_{p,p},\\
\mu_1 &= \rho_{p,1} + \binom{p-1}{1}\rho_{p,2} + \cdots + \rho_{p,p},\\
\mu_2 &= \rho_{p,2} + \cdots + \rho_{p,p},\\
&\cdots\cdots\cdots

if we put

$$\lambda_{p,m} = \binom{p}{m}\rho_{p,m} = \binom{p}{m}\Delta^{p-m}\mu_m.$$

The right side of equation (9) may be written as a Stieltjes integral if we introduce with Hausdorff the function $\chi_p(t)$ defined as follows: $\chi_p(t)$ is a step-function defined for $0 \leqq t \leqq 1$, with $\chi_p(0) = 0$, its points of discontinuity being for the values $t = m/p$ $(m = 0, 1, 2, \cdots, p)$, where its jump has the value $\lambda_{p,m}$. This function will be completely defined by the additional condition $2\chi_p(t) = \chi_p(t+0) + \chi_p(t-0)$ for $0 < t < 1$. From the first equation (7) we get $\chi_p(1) = \mu_0$ and the conditions (8) show that $\chi_p(t)$ is a non-decreasing function. From (9) we obtain

$$(10) \quad \mu_n = \int_0^1 \frac{t\left(t-\frac{1}{p}\right)\left(t-\frac{2}{p}\right)\cdots\left(t-\frac{n-1}{p}\right)}{\left(1-\frac{1}{p}\right)\left(1-\frac{2}{p}\right)\cdots\left(1-\frac{n-1}{p}\right)} d\chi_p(t)$$

$$= \int_0^1 t^n d\chi_p(t) + O\left(\frac{1}{p}\right).$$

The remainder of the proof is similar to Hausdorff's. A theorem of E. Helly* insures the existence of a sub-sequence $\chi_q(t)$ which converges to a monotonic function $\bar{\chi}(t)$. When $p = q \to \infty$, equation (10) reduces to (1) with $\chi(t) = \bar{\chi}(t)$. The relation $\int_0^1 t^n d\chi_q(t) \to \int_0^1 t^n d\bar{\chi}(t)$ follows from a theorem of Helly (loc. cit. pp. 288–289) or just as well from a known theorem on Riemann integrals as follows:

$$\int_0^1 t^n d\chi_q(t) = \chi_q(1) - n\int_0^1 \chi_q(t)t^{n-1}dt$$

$$\to \bar{\chi}(1) - n\int_0^1 \bar{\chi}(t)t^{n-1}dt = \int_0^1 t^n d\bar{\chi}(t).$$

This proves the first part of the theorem. Using the fact that $\chi(t)$ is uniquely defined at all its points of continuity by the set (1) and the condition $\chi(0) = 0$, Hausdorff proved (loc. cit., pp. 225–226) that the whole sequence $\chi_p(t)$ converges to a monotonic function $\chi(t)$ which solves the problem of moments.

* E. Helly, *Über lineare Funktionaloperationen*, Sitzungsberichte der Wiener Akademie, vol. 121 IIa (1912), p. 286.

The second statement of the theorem concerning the solution of bounded variation is also readily proved. If we suppose the function $\chi(t)$ of equations (1) to be of bounded variation, then $\chi(t)=\chi^1(t)-\chi^2(t)$, with $\chi^1(t)$ and $\chi^2(t)$ both non-decreasing functions. In Hausdorff's notation (loc. cit., p. 231) we have $\lambda_{p,m}=\lambda_{p,}{}^1{}_m-\lambda_{p,}{}^2{}_m$ and hence

$$\sum_{m=0}^{p}|\lambda_{p,m}| \leqq \sum_{m=0}^{p}\lambda_{p,m}^{1} + \sum_{m=0}^{p}\lambda_{p,m}^{2} = \mu_0{}^1 + \mu_0{}^2,$$

which proves the necessity of the condition (3). Conversely, if (3) or $\sum_{m=0}^{p}|\lambda_{p,m}|<K$ holds for $p\to\infty$, then the sequence $\chi_p(t)$ is a sequence of uniformly bounded functions of uniformly bounded variations. Another theorem of E. Helly* insures the existence of a sub-sequence $\chi_q(t)$ converging to a function $\bar{\chi}(t)$ of bounded variation. Just as above, on making $p=q\to\infty$, equation (10) becomes (1) with $\chi(t)=\bar{\chi}(t)$ and the second part of the theorem is proved. Moreover, formula (10) and the same arguments used by Hausdorff for the previous case permit us to prove that the whole sequence $\chi_p(t)$ converges to a function $\chi(t)$ of bounded variation satisfying the equations (1).

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* E. Helly, loc. cit., §6, Theorem 7 on p. 283. This theorem is a corollary of his first theorem used above.

ON SURFACES IN SPACE OF r DIMENSIONS

BY B. C. WONG

Consider a surface F^n of order n in r-space. Let it be the complete intersection of $q \leqq r-2$ varieties $V^{n_1}_{k_1}, V^{n_2}_{k_2}, \cdots, V^{n_q}_{k_q}$ of orders $n_1, n_2, \cdots, n_q$ and of dimensions $k_1, k_2, \cdots, k_q$, respectively, where

$$\text{(A)} \qquad \begin{aligned} &3 \leqq k_1, k_2, \cdots, k_q \leqq r-2, \\ &k_1 + k_2 + \cdots + k_q = r(q-1) + 2. \end{aligned}$$

Project F^n on an S_3. The projection F'^n has a number of characteristics of which we note the following six: n, its order; a, the order of its tangent cone; b, the order of its double curve; j, the number of its pinch-points; t, the number of its triple points; and m, its class. If we project F^n on an S_4, the projection has a finite number, d, of improper double points. We shall call these seven characteristics, of which n, a, t, m are often regarded as essential, the characteristics of F^n, and they are known to satisfy the following relations:*

$$\text{(B)} \qquad \begin{aligned} &a + 2b = n(n-1), \quad j + 2d = n(n-1) - a, \\ &j = \tfrac{1}{4}\left[a(3n-4) - n(n-1)(n-2) + 6t - 2m\right], \\ &d = \tfrac{1}{8}\left[n(n-1)(n+2) - 3an - 6t + 2m\right]. \end{aligned}$$

For $r=5$, $q=3$, $k_1=k_2=k_3=4$, F^n is the intersection of three hypersurfaces in S_5. Formulas for its characteristics are known† and they are symmetric functions of the orders of the hypersurfaces. In this note we present analogous formulas for the same characteristics of F^n for r general and for $q \leqq r-2$. As the method of obtaining these formulas is familiar and has been applied by the writer time and again to similar enumerative problems,‡ we shall here omit all demonstration.

---

* Severi, *Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni, e a'suoi punti tripli apparenti*, Rendiconti di Palermo, vol. 15 (1901), pp. 33–51.

† B. C. Wong, *On surfaces in spaces of four and five dimensions*, this Bulletin, vol. 36 (1930), pp. 681–686. Opportunity is here taken to correct an error in the formula for T'' on page 685 of this paper. The formula should read

$$T'' = \tfrac{1}{2}\lambda\mu\nu(\lambda-1)(\mu-1)(\nu-1)(\mu\nu+\nu\lambda+\lambda\mu-2\lambda-2\mu-2\nu).$$

‡ B. C. Wong, loc. cit., and also the paper *On the number of apparent double points of r-space curves*, this Bulletin, vol. 37 (1931), pp. 421–423.

If $q=r-2$, and, from (A), $k_1=k_2=\cdots=k_{r-2}=r-1$, F^n is the complete intersection of $r-2$ hypersurfaces in S_r. The formulas for its characteristics are

$$
\begin{aligned}
n &= n_1n_2\cdots n_{r-2},\\
a &= n(\textstyle\sum n_i - r + 2),\\
b &= \tfrac{1}{2}n(n - \textstyle\sum n_i + r - 3),\\
d &= \tfrac{1}{2}n[n - \textstyle\sum n_in_j + (r-4)\sum n_i - \tfrac{1}{2}(r-3)(r-4)],\\
j &= n[\textstyle\sum n_in_j - (r-3)\sum n_i + \tfrac{1}{2}(r-2)(r-3)],\\
t &= \tfrac{1}{6}n[n(n-3\textstyle\sum n_i) + 3(r-3)(n-2\sum n_i)\\
&\quad + 2(\textstyle\sum n_i^2 + 3\sum n_in_j) + (r-3)(3r-8)],\\
m &= n[\textstyle\sum(n_i-1)^2 + \sum(n_i-1)(n_j-1)], \qquad (i\neq j).
\end{aligned}
$$

Now if $q\leqq r-2$, one or more of the k's will be less than $r-1$. Let the ith variety $V^{n_i}_{k_i}$ be intersected by a general S_{r+2-k_i} in a surface F^{n_i}. We assume known the characteristics a_i, b_i, t_i of F^{n_i} besides n_i. The characteristics of F^n are given by the following formulas which are functions of n_i and q, and also of a_i, b_i and t_i:

$$
\begin{aligned}
n &= n_1n_2\cdots n_q,\\
a &= n(\textstyle\sum n_i - q) - 2n\sum b_i/n_i = n\sum a_i/n_i,\\
b &= \tfrac{1}{2}n(n-\textstyle\sum n_i + q - 1) + \sum b_i/n_i = \tfrac{1}{2}n(n-1) - \tfrac{1}{2}n\textstyle\sum a_i/n_i,\\
d &= \tfrac{1}{2}n[n - \textstyle\sum n_in_j + (q-2)\sum n_i - \tfrac{1}{2}(q-1)(q-2)]\\
&\quad + n\textstyle\sum_i(\sum n_j - q + 1)b_i/n_i - 2n\sum b_ib_j/n_in_j,\\
j &= n[\textstyle\sum n_in_j - (q-1)\sum n_i + \tfrac{1}{2}q(q-1)]\\
&\quad - 2n\textstyle\sum_i(\sum n_j - q)b_i/n_i + 4n\sum b_ib_j/n_in_j,\\
t &= \tfrac{1}{6}n[n(n-3\textstyle\sum n_i) + 3(q-1)(n-2\sum n_i)\\
&\quad + 2(\textstyle\sum n_i^2 + 3\sum n_in_j) + (q-1)(3q-2)] + n\sum t_i/n_i\\
&\quad + n\textstyle\sum_i[n - 2\sum n_j - n_i + 2(q-1)]b_i/n_i + 4n\sum b_ib_j/n_in_j,\\
m &= n[\textstyle\sum(n_i-1)^2 + \sum(n_i-1)(n_j-1)] + 3n\sum t_i/n_i\\
&\quad - n\textstyle\sum_i(2\sum n_j + 3n_i - 2q + 2)b_i/n_i + 4\sum b_ib_j/n_in_j,\\
&\qquad\qquad (i\neq j).
\end{aligned}
$$

All the formulas of each of these two sets satisfy relations (B).

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ON THE COMPACTNESS OF THE SPACE L_p

BY J. D. TAMARKIN*

1. *Introduction.* Let R_n be the n-dimensional euclidean space and $L_p(p>1)$ the function-space consisting of all the functions $f(x)$ defined and measurable over R_n, and such that the integrals

$$\int_{R_n} |f(x)|^p dx < \infty,$$

while the metric of the space L_p is defined, as usual, by the "distance"

$$\|f\| \equiv \left[\int_{R_n} |f(x)|^p dx\right]^{1/p}.$$

All the notions of *boundedness, convergence, limits, approximations*, etc., used in this note will be relative to this metric, unless explicitly specified to the contrary.

Let $S(x, \epsilon)$ be the n-dimensional sphere with center at x and radius ϵ. We designate by $V(\epsilon)$ the volume of $S(x, \epsilon)$, and by

$$f_\epsilon(x) \equiv \frac{1}{V(\epsilon)} \int_{S(x,\epsilon)} f(y)dy \tag{1}$$

the *moving average* of $f(x)$. Finally, let C_N be the n-dimensional cube, with center at the origin and edges of length $2N$ parallel to the coordinate axes. We set

$$f^N(x) = \begin{cases} f(x) \text{ when } x \text{ is in } C_N, \\ 0 \text{ elsewhere.} \end{cases} \tag{2}$$

A set $\mathfrak{F}$ of elements of L_p is called *compact* if every subset of $\mathfrak{F}$ contains at least one convergent sequence. In a recent paper Kolmogoroff† has derived necessary and sufficient conditions in order that $\mathfrak{F}$ be compact, under the restriction that all the elements of $\mathfrak{F}$ vanish outside of a fixed bounded measurable point

* Presented to the Society, October 31, 1931.

† Göttinger Nachrichten, 1931, pp. 60–63.

set F of R_n. In order that a set $\mathfrak{F}$ of this type be compact it is necessary and sufficient that there exist a fixed positive constant K and, for each given positive δ, a positive $\epsilon=\epsilon(\delta)$ depending only on δ and $\mathfrak{F}$, such that for all elements of $\mathfrak{F}$,

$$\int_F |f(x)|^p dx \leqq K, \tag{3}$$

$$\int_F |f(x) - f_\epsilon(x)|^p dx \leqq \delta. \tag{4}$$

In the case where F is not bounded, in particular where $F=R_n$, the conditions above, being still necessary, are no longer sufficient for the compactness of $\mathfrak{F}$. This can be shown by the following simple example ($n=1$). Let

$$\mathfrak{F} \equiv \{f_\nu(x)\}, \quad (\nu = 0, \pm 1, \pm 2, \cdots), \tag{5}$$

where $f_\nu(x)=1$ for $\nu \leqq x < \nu+1$, and 0 elsewhere. It is plain that $\|f_i - f_j\| = 2^{1/p}$, $i \neq j$, so that no subsequence of $\mathfrak{F}$ can converge. On the other hand we have, for all values of ν,

$$\|f_\nu\| = 1,$$

$$f_\nu(x) = f_0(x-\nu) = \frac{1}{2\epsilon}\int_{x-\nu-\epsilon}^{x-\nu+\epsilon} [f_0(x-\nu) - f_0(\xi)]d\xi,$$

$$\|f_\nu - (f_\nu)_\epsilon\|^p = \int_{-\infty}^{\infty} dx \left| \frac{1}{2\epsilon}\int_{x-\nu-\epsilon}^{x-\nu+\epsilon} [f_0(x-\nu) - f_0(\xi)]d\xi \right|^p$$

$$= \|f_0 - (f_0)_\epsilon\|^p,$$

and so the distance $\|f_\nu - (f_\nu)_\epsilon\|$ can be made arbitrarily small by taking ϵ sufficiently small, uniformly in ν. Hence the conditions above are satisfied but $\mathfrak{F}$ is not compact. The purpose of the present note is to derive a set of necessary and sufficient conditions for the compactness of any set $\mathfrak{F} \subset L_p$. This can be easily done by introducing some slight modifications in Kolmogoroff's arguments.

2. *Conditions for Compactness.* The conditions for compactness of an arbitrary set $\mathfrak{F} \subset L_p$ are given by the following theorem.

THEOREM. *In order that a set $\mathfrak{F} \subset L_p (p>1)$ be compact it is necessary and sufficient that $\mathfrak{F}$ be bounded, and that the elements $f(x)$ of $\mathfrak{F}$ should be approximated, arbitrarily closely and uniformly over $\mathfrak{F}$, by the corresponding functions f_h and f^N. In other words, there should exist a constant $F_0 \equiv F_0(\mathfrak{F})$, and, for a given positive δ, two constants, $\epsilon \equiv \epsilon(\delta; \mathfrak{F})$ and $N_0 \equiv N_0(\delta; \mathfrak{F})$, depending only on $\mathfrak{F}$, or on δ and $\mathfrak{F}$ respectively, but not on $f(x) \subset \mathfrak{F}$, such that the following conditions are satisfied:*

(i) $$\|f\| \leqq F_0;$$

(ii) $$\|f - f_h\| \leqq \delta, \ 0 < h \leqq \epsilon;$$

(iii) $$\|f - f^N\| \leqq \delta, \ N \geqq N_0;$$

for all elements $f \subset \mathfrak{F}$.

The proof of this theorem is based upon several simple lemmas which are mostly well known and are stated here merely for the convenience of the reader.

LEMMA 1. *If $f, g \subset L_p (p \geqq 1)$, then*

(6) $$\|f_h - g_h\| \leqq \|f - g\|,$$

(7) $$\|f^N - g^N\| \leqq \|f - g\|.$$

In particular,

(8) $$\|f_h\| \leqq \|f\|, \ \|f^N\| \leqq \|f\|.$$

LEMMA 2. *For a fixed element $f \subset L_p (p \geqq 1)$,*

(9) $$\|f - f_h\| \to 0 \ \text{as} \ \ h \to 0,$$

(10) $$\|f - f^N\| \to 0 \ \text{as} \ \ N \to \infty.$$

LEMMA 3. *Under the assumption of Lemma 2,*

(11) $$\|f^N - (f^N)_h\| \leqq 2\|f - f^N\| + \|f - f_h\|.$$

To simplify the notation we shall consider only the one-dimensional case, so that $-\infty < x < \infty$ and

(12) $$\|f\| \equiv \left[\int_{-\infty}^{\infty} |f(x)|^p dx \right]^{1/p}.$$

The definition of $f_h(x)$ will be slightly modified by setting

$$(13) \qquad f_h(x) \equiv h^{-1}\int_x^{x+h} f(t)dt = h^{-1}\int_0^h f(x+t)dt,$$

while, as before,

$$(14) \qquad f^N(x) = f(x), \text{ or } 0, \text{ according as } |x| \leqq N, \text{ or } > N.$$

The proofs of the lemmas above will be only sketched.

Formula (7) follows directly from the definition of $f^N(x)$, $g^N(x)$. Formula (6) is derived by using Hölder's inequality,

$$|f_h(x) - g_h(x)|^p \leqq h^{-1}\int_0^h |f(x+t) - g(x+t)|^p dt,$$

$$\int_{-\infty}^{\infty} |f_h(x) - g_h(x)|^p dx \leqq h^{-1}\int_0^h dt \int_{-\infty}^{\infty} |f(x+t) - g(x+t)|^p dx = \|f-g\|^p,$$

when $p>1$, and even more simply when $p=1$. Formula (10) is obvious, while (9) follows from

$$|f(x) - f_h(x)|^p \leqq h^{-p}\left[\int_0^h |f(x) - f(x+t)|\, dt\right]^p \leqq h^{-1}\int_0^h |f(x) - f(x+t)|^p dt,$$

and from a known theorem* according to which

$$\int_{-\infty}^{\infty} |f(x) - f(x+t)|^p dx \to 0 \text{ as } t \to 0,\ f \subset L_p.$$

To prove (11) we observe that, by (6),

$$\|f^N - (f^N)_h\| \leqq \|f^N - f\| + \|f - f_h\| + \|f_h - (f^N)_h\| \leqq 2\|f - f^N\| + \|f - f_h\|.$$

* Hobson, *Theory of Functions of a Real Variable*, vol. I, 3d ed., 1927, pp. 636–639. The result is stated and proved there only for $p=1$ and $p=2$, but can be extended readily to the general case of any $p \geqq 1$.

We now turn to the proof of our theorem. The necessity of conditions (i) and (ii) is established in precisely the same fashion as in the case of Kolmogoroff. To prove the necessity of (iii) we shall use an analogous argument. If (iii) is not satisfied, we can find a fixed positive δ and two sequences, of functions $\{f_n(x)\} \subset \mathfrak{F}$, and of positive numbers $\{N_n\} \to \infty$ as $n \to \infty$, such that, on setting

$$f_n' \equiv (f_n)^{N_n},$$

we would have

$$\|f_n - f_n'\| > \delta.$$

Hence, no matter what is $f(x) \subset L_p$, by (7),

$$\begin{aligned} \delta < \|f_n - f_n'\| &\leqq \|f_n - f\| + \|f - f^{N_n}\| + \|f^{N_n} - f_n'\| \\ &\leqq 2\|f - f_n\| + \|f - f^{N_n}\|. \end{aligned}$$

On the other hand, since $N_n \to \infty$, we have by (10)

$$\|f - f^{N_n}\| \to \infty.$$

The conclusion is that

$$\lim_{n \to \infty} \|f - f_n\| \geqq \delta/2,$$

and so, no subsequence of $\{f_n(x)\}$ can converge. This shows that $\mathfrak{F}$ is not compact.

We pass on to the proof of the sufficiency of the conditions (i)–(iii). Assume that these conditions are satisfied. Let $\{f_n(x)\}$ be an arbitrary sequence of $\mathfrak{F}$. To abbreviate we put

$$f_n(x) \equiv \phi(x), \quad f_m(x) \equiv \psi(x),$$

where n and m are arbitrary positive integers. According to (ii) and (iii) a positive h can be fixed so small and a positive N so large that, for a given $\delta > 0$,

$$\|\phi - \phi^N\|, \quad \|\psi - \psi^N\|, \quad \|\phi - \phi_h\|, \quad \|\psi - \psi_h\| \leqq \delta.$$

By (i), (8), and (11),

$$\|\phi^N\| \leqq F_0, \quad \|\phi^N - (\phi^N)_h\| \leqq 3\delta,$$

whence, according to the result of Kolmogoroff, the set $\{\phi\}$ is compact. Hence a sequence of subscripts ν_s can be selected in

such a way that when n and $m \to \infty$ assuming the values belonging to this sequence, then

$$\|(\phi^N)_h - (\psi^N)_h\| \to 0.$$

In particular, for n and m sufficiently large,

$$\|(\phi^N)_h - (\psi^N)_h\| \leqq \delta.$$

On combining these inequalities we get, with reference to (11),

$$\|f_n - f_m\| \leqq \|\phi - \phi^N\| + \|\phi^N - (\phi^N)_h\| + \|(\phi^N)_h - (\psi^N)_h\| + \|(\psi^N)_h - \psi^N\| + \|\psi^N - \psi\| \leqq 9\delta.$$

This shows that an arbitrary sequence of $\mathfrak{F}$ contains a convergent subsequence, hence $\mathfrak{F}$ is compact.

3. *Conclusion.* It is interesting to observe that the conditions (i)–(iii) of our theorem are independent, in the sense that no two of them imply the third.

(a) That conditions (i) and (ii) do not imply (iii) is shown by the example of the set (5) which satisfies (i) and (ii) but not (iii), for

$$\|f_n - (f_n)^N\| = 1, \qquad (0 < N < n).$$

(b) That (i) and (iii) do not imply (ii) can be shown by the example of any set of functions which vanish outside of a fixed finite interval (a, b), while the set is orthonormal for this interval.

(c) To prove that (ii) and (iii) do not imply (i) it suffices to consider the set $[f_1, f_2, \cdots, f_n, \cdots]$ defined by the condition that $f_n(x) = f(x) + n$ on (0, 1), and 0 elsewhere, $f(x)$ being an arbitrary function $\subset L_p$ and vanishing outside of (0,1).

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CONCERNING ADJUNCTIONS TO ALGEBRAS

BY J. L. DORROH*

In §3 of his paper, *Algebras which do not possess a finite basis*,† J. H. M. Wedderburn gives a set of postulates and definitions for an algebra. The question of the possibility of adjoining an identity to such an algebra arises. The purpose of the present paper is to show that this adjunction is always possible.

It may be seen from the arguments that Theorems 1 and 2 remain true if the term *algebra* be replaced by one implying a set of elements of which it is assumed only that it is an abelian group under addition and a semi-group under multiplication. The proof of Theorem 3 employs a distributive property, as will be indicated.

THEOREM 1.‡ *If A is an algebra, then there exists an algebra B which contains a proper, invariant subalgebra A' isomorphic§ with A and an element I not in A' such that, for every element b of B, the relation $Ib=bI=b$ holds.*

If x is an element of A and n is a positive rational integer, let $nx=xn$ denote the sum $x+x+\cdots+x$ (n summands) and let $(-n)x=x(-n)$ denote the same sum as $n(-x)$. Let $0\cdot x=x\cdot 0=0$.

Let B denote the set of all pairs (n, x) where n is a rational integer and x is an element of A. Suppose that $(n, x)+(m, y)=(n+m, x+y)$. Then B is an abelian group under addition. Let $(n, x)\ (m, y)=(nm, ny+mx+xy)$. Then B is an algebra and the subset A' of B consisting of all the elements of B of the form $(0, x)$ is isomorphic with A and is invariant in B. Let I denote the element $(1, 0)$ of B; $(1, 0)\ (n, x)=(n, x)\ (1, 0)=(n, x)$ for every n and x.

The multiplication in B is distributive, associative and distributive, commutative, respectively, provided the same is true for A.

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† Transactions of this Society, vol. 26 (1924), pp. 395–426.

‡ For a proof of this theorem for an algebra with a finite basis, see, for example, L. E. Dickson, *Algebras and their Arithmetics*, 1923, p. 97.

§ In this paper isomorphism means *simple* isomorphism.

THEOREM 2. *If an algebra A has the properties*

(1) *if x is an element of A and m is a positive rational integer, A contains an element y such that $my=x$;*

(2) *if x is an element of A and m is a positive rational integer and $x\neq 0$, we have $mx\neq 0$; then there exists an algebra B which contains a proper, invariant subalgebra isomorphic with A and a proper subalgebra isomorphic with the rational field.*

If x is an element of A and m is a positive integer, it follows from (1) that A contains an element y such $my=x$; let y be denoted by x/m; it follows from (2) that x/m is unique.

If r is a rational number, let $r=m/n$, where m and n are integers and n is positive. Let $rx=xr=m(x/n)$ for each element x of A. Then rx is unique, for suppose that m and n have been chosen relatively prime and let k denote a positive integer, then $km(x/(kn))=m(x/n)$, for $kn[km(x/(kn))]=kmx=kn[m(x/n)]$ and the equality of the brackets follows from (2).

Let B denote the set of all pairs (r, x) where r is a rational number and x is an element of A. Under the following rules for addition and multiplication B is an algebra:

$$(1) \qquad (r, x)+(r', x')=(r+r', x+x'),$$

$$(2) \qquad (r, x)(r', x')=(rr', rx'+r'x+xx').$$

The element $(1, 0)$ of B is its identity, since we have $(1, 0)\ (r, x)=(r, x)=(r, x)\ (1, 0)$.

The set of all the elements of B of the form $(0, x)$ is a proper, invariant subalgebra of B and it is isomorphic with A.

The set of all the elements of B of the form $(r, 0)$ is a proper subalgebra of B isomorphic with the rational field.

The multiplication in B is distributive, associative and distributive, commutative, respectively, provided the same is true for A.

THEOREM 3. *If A is an algebra subject to condition (2) of Theorem 2, there exists an algebra B which contains a subalgebra isomorphic with A and such that, if b is an element of B and n is a positive rational integer, B contains a unique element c such that $nc=b$.*

In the set D of all pairs (r, x), where r is a rational number and x is an element of A, (r, x) will be said to be equivalent to (r', x') if $svx=tux'$, where $r=s/t$ and $r'=u/v$, and where s, t, u

and v are integers and t and v are positive. This relation will be denoted by $(r, x)\sim(r', x')$. From the hypothesis concerning A it follows that the equivalence of (r, x) and (r', x') is independent of the choice of rational expressions for r and r'.

(1) $(r, x)\sim(r, x)$.

(2) If $(r, x)\sim(r', x')$, then $(r', x')\sim(r, x)$.

(3) If $(r, x)\sim(r', x')$ and $(r', x')\sim(r'', x'')$, then also $(r, x)\sim(r'', x'')$. Let $r=s/t$, $r'=u/v$, $r''=p/q$; s, t, u, v, p, q integers and $t>0$, $v>0$, $q>0$. If $svx=tux'$ and $uqx'=vpx''$, then $svqx=tuqx'$ and $tuqx'=tvpx''$; hence $svqx=tvpx''$ and therefore $sqx=tpx''$.

Let $(r, x)+(r', x')=(1/(tv), svx+tux')$, where $r=s/t$, $r'=u/v$; s, t, u, v integers and t, v positive. If $(r', x')\sim(r'', x'')$, then $(r, x)+(r', x')\sim(r, x)+(r'', x'')$. Let $r''=p/q$; p, q integers and $q>0$. Then $(r, x)+(r'', x'')=(1/(tq), sqx+ptx'')$. By hypothesis $uqx'=vpx''$. Hence we have $tuqx'=tvpx''$ and therefore $(1/(tvq), svqx+tuqx')\sim(1/(tvq), svqx+tvpx'')$, but $(1/(tvq), svqx+tuqx')\sim(1/(tv), svx+tux')$ and $(1/(tvq), svqx+tvpx'')\sim(1/(tq), sqx+tpx'')$. It follows that if $(r, x)\sim(r''', x''')$ and $(r', x')\sim(r'', x'')$, then $(r, x)+(r', x')\sim(r''', x''')+(r'', x'')$.

Let $(r, x)(r', x')=(rr', xx')$. Then if $(r', x')\sim(r'', x'')$, $(r, x)(r', x')\sim(r, x)(r'', x'')$. For by hypothesis $uqx'=vpx''$ and hence $uqxx'=vpxx''$ ($r=s/t$, $r'=u/v$, $r''=p/q$ as before), and hence* $stuqxx'=stvpxx''$. Similarly $(r', x')(r, x)\sim(r'', x'')(r, x)$. Hence if $(r, x)\sim(r''', x''')$, then $(r, x)(r', x')\sim(r''', x''')\cdot(r'', x'')$.

For each rational number w let $w(r, x)=(r, x)w=(wr, x)$.

For each (r, x) of D let $C_{(r,x)}$ denote the class of all pairs (r', x') of D such that $(r', x')\sim(r, x)$. Then $C_{(r,x)}=C_{(r'',x'')}$ if, and only if, $(r, x)\sim(r'', x'')$. Let $C_{(r,x)}+C_{(r',x')}=C_{(r,x)+(r',x')}$ and let $C_{(r,x)}C_{(r',x')}=C_{(r,x)(r',x')}$.† These products and sums are unique. For each rational number w let $wC_{(r,x)}=[C_{(r,x)}]w=C_{w(r,x)}$.

* To establish this equation and the corresponding one implied by the succeeding expressions in the text, it is sufficient to assume that if n is a positive rational integer and x and y are elements of A, then $x(ny)=nxy$ and $(ny)x=n(yx)$. That this is a distributive property is seen from the definition of ny.

† For convenience the ordinary symbols for addition and multiplication are retained, although the operations are not the ones they would ordinarily indicate for classes.

Let B denote the collection of all the classes $C_{(r,x)}$. Then B is an algebra. The subset A' of B consisting of all the classes $C_{(1,x)}$ of B is isomorphic with A and is identical with B if, and only if, for each element x of A and each positive rational integer m, A contains an element x/m such that $m(x/m)=x$.

If $C_{(r,x)}$ is an element of B and m is a positive rational integer, then $C_{(r/m,x)}$ is an element of B and $mC_{(r/m,x)}=C_{(r,x)}$.

Multiplication in B is associative, commutative, distributive, respectively, if the same is true for A.

PRINCETON UNIVERSITY

ON THE RANK OF THE PRODUCT OF CERTAIN SQUARE MATRICES*

BY W. O. MENGE

1. *Introduction.* This paper presents several theorems which were found during an investigation conducted by the author into the structure of matrices which transform given matrices into their so-called classic and rational canonical forms.† When the elementary divisors of a given matrix are known, these theorems completely determine the rank of a product of matrices of the form

$$\prod_{i=1}^{\omega}(A-\lambda_i I)^{k_i}.$$

An interesting proof of the Hamilton-Cayley theorem and a determination of the equation of minimum degree satisfied by a matrix are obtained from this point of view.

2. *Invariant Factors.* Consider the square matrix $A=(a_{ij})$ of order n with constant elements. If the n-rowed identity matrix be denoted by I, the characteristic matrix $(A-\lambda I)$ is defined as the matrix obtained by subtracting the variable λ from each principal diagonal element of A. The determinant, $D(\lambda)$, of the characteristic matrix $(A-\lambda I)$ is called the characteristic de-

* Presented to the Society, December 30, 1930. The author wishes to acknowledge his appreciation to J. A. Nyswander, University of Michigan, for many helpful suggestions throughout the progress of the work.

† Dickson, *Modern Algebraic Theory*, Chap. 5.

terminant of A. In order to secure complete generality in the subsequent discussion, the characteristic equation $D(\lambda)=0$ is assumed to have the distinct roots $\lambda_1, \lambda_2, \cdots, \lambda_\omega$ of multiplicities $n^{(1)}, n^{(2)}, \cdots, n^{(\omega)}$, respectively. Hence

$$D(\lambda) = (\lambda - \lambda_1)^{n^{(1)}}(\lambda - \lambda_2)^{n^{(2)}} \cdots (\lambda - \lambda_\omega)^{n^{(\omega)}},$$

where $\sum_{i=1}^{\omega} n^{(i)} = n$.

Designate by $G_j(\lambda)$ the highest common factor of all jth minors (of order $n-j$) of the characteristic determinant $D(\lambda)$. These common factors are chosen so that the coefficient of the highest power of λ is unity. Thus one can write

$$G_j(\lambda) = (\lambda - \lambda_1)^{m_j^{(1)}}(\lambda - \lambda_2)^{m_j^{(2)}} \cdots (\lambda - \lambda_\omega)^{m_j^{(\omega)}}, \quad (j = 1, 2, \cdots, n).$$

The invariant factors of the matrix A are defined as

$$D_j(\lambda) = G_{j-1}(\lambda)/G_j(\lambda) = (\lambda - \lambda_1)^{n_j^{(1)}}(\lambda - \lambda_2)^{n_j^{(2)}} \cdots (\lambda - \lambda_\omega)^{n_j^{(\omega)}}$$
$$= \lambda^{n_j} - d_{1j} - d_{2j}\lambda - \cdots - d_{n_j,j}\lambda^{n_j-1}, \quad (j = 1, 2, \cdots, n),$$

where

$$G_0(\lambda) = D(\lambda), \qquad n_j^{(i)} = m_{j-1}^{(i)} - m_j^{(i)}, \qquad n_j = \sum_{i=1}^{\omega} n_j^{(i)},$$
$$(i = 1, 2, \cdots, \omega; j = 1, 2, \cdots, n).$$

3. *Classic Canonical Form.* Define the integer σ, so that

$$D_j(\lambda) \not\equiv 1 \text{ for } j \leqq \sigma; \quad D_j(\lambda) = 1 \text{ for } j > \sigma.$$

The classic canonical form for the n-rowed square matrix A is defined as

$$P = \begin{vmatrix} N_{11} & 0 & 0 & \cdots 0 \\ 0 & N_{21} & 0 & \cdots 0 \\ 0 & 0 & N_{31} & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots N_{\omega\sigma} \end{vmatrix},$$

where N_{ij} is an $n_j^{(i)}$-rowed square matrix of the form

$$N_{ij} = \begin{vmatrix} \lambda_i & 0 & 0 & \cdots 0 \\ 1 & \lambda_i & 0 & \cdots 0 \\ 0 & 1 & \lambda_i & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots \lambda_i \end{vmatrix}, \qquad \begin{matrix} (i = 1, 2, \cdots, \omega; \\ j = 1, 2, \cdots, \sigma). \end{matrix}$$

Each of the matrices N_{ij} is associated with a particular elementary divisor. From the above definition it can be easily verified that the invariant factors and elementary divisors of the matrix P are identical with the invariant factors and elementary divisors of the matrix A.

4. *Ranks.* It has been shown* that a necessary and sufficient condition that a non-singular matrix B exist such that

$$A = B \cdot P \cdot B^{-1}$$

is that the matrices A and P have the same invariant factors, or if we prefer, the same elementary divisors. It follows immediately that a non-singular matrix B exists such that

$$A - \lambda I = B \cdot (P - \lambda I) \cdot B^{-1}. \tag{1}$$

Since equation (1) is an identity in the variable λ, one can write

$$\begin{aligned} \prod_{i=1}^{\omega} (A - \lambda_i I)^{k_i} &= B^m \cdot \prod_{i=1}^{\omega} (P - \lambda_i I)^{k_i} \cdot (B^{-1})^m \\ &= B^m \cdot \prod_{i=1}^{\omega} (P - \lambda_i I)^{k_i} \cdot (B^m)^{-1}, \end{aligned} \tag{2}$$

where $m = \sum_{i=1}^{\omega} k_i$. Since the matrix B is non-singular it follows that B^m is also non-singular. Hence we may state the following lemma.

LEMMA 1. *The rank of the product*

$$\prod_{i=1}^{\omega} (A - \lambda_i I)^{k_i}$$

is identical with the rank of the product

$$\prod_{i=1}^{\omega} (P - \lambda_i I)^{k_i}.$$

It will be seen that the rank of any given product of the latter type can be determined by the invariant factors and elementary divisors of the matrix A.

Using the notation of Kronecker, define a matrix of order q, $(\delta_{r,s+1}) = (c_{rs})$, where $c_{rs} = 1$ for $r = s+1$, $c_{rs} = 0$ for $r \neq s+1$. It is easy to verify that $(\delta_{r,s+1})^k = (\delta_{r,s+k})$, and to prove the following result.

* Bôcher, *Introduction to Higher Algebra*, pp. 279–283.

LEMMA 2. *The matrix* $(\delta_{r,s+k})$ *is of rank* $q-k$ *if* $q-k \geqq 0$ *and of zero rank if* $q-k<0$.

In forming the product $(P-\lambda_i I)^{k_i}$ from its linear factors it is apparent from the special nature of each factor that the determinants $|N_{kj}-\lambda_i I|$ associated with each elementary divisor multiply each other independently of the determinants associated with other elementary divisors. The product is of the same form as each factor, and the determinant associated with each elementary divisor in the product is made up of the product of the corresponding determinants in the several factors. The determinants $|N_{ij}-\lambda_i I|$ are of the form $(\delta_{r,s+1})$ with $q=n_j^{(i)}$, and the determinants $|N_{kj}-\lambda_i I|$ with $h \neq i$ are non-singular.

Define $r_i^{(l)}$ as the number of elementary divisors of A divisible by $(\lambda-\lambda_i)^l$. Then $r_i^{(l)}$ will also be the number of elementary divisors of P which contain the factor $(\lambda-\lambda_i)^l$. Upon applying Lemma 2 for the multiplication of matrices of this form it is seen that the rank of the determinant associated with the elementary divisor $(\lambda-\lambda_i)^{n_j^{(i)}}$ in the product $(P-\lambda_i I)^{k_i}$ will be $n_j^{(i)}-k_i$ if $n_j^{(i)}-k_i \geqq 0$ and will be zero if $n_j^{(i)}-k_i<0$. The determinants associated with elementary divisors which do not contain the root λ_i will be non-singular. Thus the rank of the matrix of the product $(P-\lambda_i I)^{k_i}$ is seen to be

$$n - \sum_{l=1}^{k_i} r_i^{(l)}.$$

In the product $(P-\lambda_i I)^{k_i}(P-\lambda_j I)^{k_j}$, $j \neq i$, the determinants associated with elementary divisors involving either of the roots λ_i, λ_j are singular and of the form $(\delta_{r,s+1})$, while those associated with elementary divisors involving other roots are non-singular. Hence the rank of the product $(P-\lambda_i I)^{k_i}(P-\lambda_j I)^{k_j}$ is

$$n - \sum_{l=1}^{k_i} r_i^{(l)} - \sum_{l=1}^{k_j} r_j^{(l)}.$$

In general we find that the rank of the product

$$\prod_{i=1}^{\omega}(P-\lambda_i I)^{k_i}$$

is

$$n - \sum_{i=1}^{\omega} \sum_{l=1}^{k_i} r_i^{(l)}.$$

Upon applying Lemma 1 to the result just obtained we have the following theorem.

THEOREM 1. *The rank of the product*

$$\prod_{i=1}^{\omega}(A - \lambda_i I)^{k_i}$$

is

(i) $$n - \sum_{i=1}^{\omega}\sum_{l=1}^{k_i} r_i^{(l)}.$$

This result may also be written in the form

(ii) $$n - \sum_{i=1}^{\omega}\left[k_i r_i^{(k_i)} + \sum_{n_h^{(i)}<k_i} n_h^{(i)}\right].$$

Define $D_j(A)$ as the matrix obtained by replacing the variable λ by the matrix A in the jth invariant factor $D_j(\lambda)$. Thus

$$D_j(A) = (A - \lambda_1 I)^{n_j^{(1)}}(A - \lambda_2 I)^{n_j^{(2)}} \cdots (A - \lambda_\omega I)^{n_j^{(\omega)}}.$$

Any matrix $f(A)$, where $f(\lambda)$ is of equal or lower degree in λ than $D_j(\lambda)$ can be obtained from $D_j(A)$ in one of the following ways:

(a) by replacing one or more factors, say $(A-\lambda_1 I)$, by another, say $(A-\lambda_2 I)$;

(b) by omitting one or more factors, say $(A-\lambda_1 I)$, or by replacing them by non-singular matrices.

Under Case (a), the rank R of the altered matrix is, according to Theorem 1,

$$R = n - \sum_{l=1}^{n_j^{(1)}-1} r_1^{(l)} - \sum_{l=1}^{n_j^{(2)}+1} r_2^{(l)} - \sum_{i=2}^{\omega}\sum_{l=1}^{n_j^{(i)}} r_i^{(l)}.$$

Similarily the rank R_j of the matrix $D_j(A)$ is

$$R_j = n - \sum_{i=1}^{\omega}\sum_{l=1}^{n_j^{(i)}} r_i^{(l)},$$

and the difference in rank is

$$R - R_j = r_1^{(n_j^{(1)})} - r_2^{(n_j^{(2)}+1)}.$$

It is apparent, however, from the definition of $r_i^{(l)}$ and the prop-

erty of each invariant factor being divisible by each succeeding invariant factor, that

$$r_1^{(n_j^{(1)})} \geqq j, \quad \text{while} \quad r_2^{(n_j^{(2)}+1)} < j.$$

From this it follows immediately that $R \geqq R_j + 1$ and hence the replacement in Case (a) increases the rank by at least unity.

Under Case (b) by Theorem I the rank R of the altered matrix is determined as

$$R = n - \sum_{i=1}^{(n_j^{(1)}-1)} r_1^{(i)} - \sum_{i=2}^{\omega} \sum_{l=1}^{n_j^{(i)}} r_i^{(l)},$$

from which one concludes

$$R - R_j = r_1^{(n_j^{(1)})}.$$

Hence the rank R is larger than the rank R_j of the invariant factor matrix $D_j(A)$. Thus one has established the following fact.

THEOREM 2. *The rank of any invariant factor matrix $D_j(A)$ is the minimum for all matrices $f(A)$, where $f(\lambda)$ is of the same or lower degree in λ than $D_j(\lambda)$.*

By setting $k_i = n_j^{(i)}$, $(i = 1, 2, \cdots, \omega)$, the result (ii) expressed in Theorem I can be applied to determine the rank R_j of $D_j(A)$. Here we find

$$R_j = n - \sum_{i=1}^{\omega} \left[n_j^{(i)} \cdot r_i^{(n_j^{(i)})} + \sum_{n_h^{(i)} < n_j^{(i)}} n_h^{(i)} \right], \tag{3}$$

the summation inside the bracket being with respect to h. Setting $s_j^{(i)} = r_i^{(n_j^{(i)})} - j$ in equation (3), we obtain

$$R_j = n - \sum_{i=1}^{\omega} \left[n_j^{(i)}(j + s_j^{(i)}) + \sum_{n_h^{(i)} < n_j^{(i)}} n_h^{(i)} \right],$$

and

$$R_j = n - j \cdot n_j - \sum_{i=1}^{\omega} \left[s_j^{(i)} n_j^{(i)} + \sum_{n_h^{(i)} < n_j^{(i)}} n_h^{(i)} \right], \tag{4}$$

since

$$\sum_{i=1}^{\omega} n_j^{(i)} = n_j.$$

Equation (4) may be written in the form

$$R_j = n - j \cdot n_j - \sum_{i=1}^{\omega} \sum_{h=j+1}^{\sigma} n_h^{(i)},$$

or

$$R_j = n - j \cdot n_j - \sum_{h=j+1}^{\sigma} n_h. \tag{5}$$

By virtue of the equation

$$\sum_{h=1}^{j} n_h = n - \sum_{h=j+1}^{\sigma} n_h,$$

the expression (5) may be written in the form

$$R_j = \sum_{h=1}^{j} n_h - j \cdot n_j = \sum_{h=1}^{j} (n_h - n_j).$$

THEOREM 3. *The rank R_j of $D_j(A)$ is*

$$\sum_{h=1}^{j} (n_h - n_j) = \sum_{h=1}^{j} n_h - j \cdot n_j = n - j \cdot n_j - \sum_{h=j+1}^{\sigma} n_h.$$

It should be noted that two well known properties of the matrix A follow directly from Theorem 3, namely, *Every square matrix satisfies its characteristic equation* and *The equation of minimum degree satisfied by the matrix A is the first invariant factor equated to zero.*

THEOREM 4. *In progressing from $D_{j+1}(A)$ to $D_j(A)$ by multiplying by successive linear factors, the rank decreases j for each linear factor.*

THEOREM 5. *The difference in the ranks of two consecutive invariant factor matrices, $D_{j+1}(A)$ and $D_j(A)$, is $j(n_j - n_{j+1})$.*

THE UNIVERSITY OF MICHIGAN

A TRIAD OF RULED SURFACES DEFINED BY RECIPROCAL POLARS*

BY A. F. CARPENTER

As an application of the theory of triads of ruled surfaces† in projective differential geometry, we shall consider here a system defined as follows. Let l_{yz} be any line element of a general ruled surface R_{yz}, the points P_y, P_z being the flecnodes of l_{yz}. The defining system of differential equations for R_{yz} will have the form

$$(1) \qquad \begin{aligned} y'' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + q_{21}y + q_{22}z &= 0, \end{aligned}$$

where differentiation is with respect to a parameter x and where $p_{12}' = 2q_{12}$, $p_{21}' = 2q_{21}$.

The planes osculating the flecnode curve C of R_{yz} at P_y, P_z intersect in a line $l_{\psi\phi}$, the points P_ψ, P_ϕ of which are those in which $l_{\psi\phi}$ is cut by the tangents to C at the respective points P_z, P_y. The expressions for ψ, ϕ are

$$(2) \qquad \psi = p_{12}z' + q_{12}z, \quad \phi = p_{21}y' + q_{21}y.$$

The polar reciprocal of $l_{\psi\phi}$ with respect to the linear complex which osculates R_{yz} along l_{yz} is the line $l_{\eta\theta}$, the points P_η, P_θ of which are given by the expressions

$$(3) \quad \eta = p_{12}z' + p_{12}p_{21}y + q_{12}z, \quad \theta = p_{21}y' + q_{21}y + p_{12}p_{21}z.$$

The points P_y, P_ψ, P_η are collinear, as are also the points P_z, P_ϕ, P_θ.

A set of three lines l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$, as thus defined, corresponds to each value of the parameter x and determines in this way three ruled surfaces R_{yz}, $R_{\psi\phi}$, $R_{\eta\theta}$. From (1), (2) and (3) we obtain the defining system of differential equations for this triad of ruled surfaces.‡ It is

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* Presented to the Society, June 13, 1931.

† A. F. Carpenter, *Triads of ruled surfaces*, Transactions of this Society, vol. 29 (1927), pp. 254–275. Hereafter denoted by the symbol T.

‡ T, p. 256.

$$
\begin{aligned}
&y' = -r_{21}y + \pi_{21}\phi, \quad \psi' = 3r_{12}\psi + (\alpha - p_{12})\phi - \alpha\theta,\\
&z' = -r_{12}z + \pi_{12}\psi, \quad \phi' = (\beta - p_{21})\psi + 3r_{21}\phi - \beta\eta,\\
(4)\qquad &y = \pi_{12}\pi_{21}(\eta - \psi), \quad \eta' = py - p_{12}p_{21}\alpha z + 3r_{12}\eta,\\
&z = \pi_{12}\pi_{21}(\theta - \phi), \quad \theta' = -p_{12}p_{21}\beta y - pz + 3r_{21}\theta,
\end{aligned}
$$

where*

$$
\begin{aligned}
&p_{12}^2 p_{21}\alpha = p_{12}^2 q_{22} - p_{12}q_{12}' + 3q_{12}^2,\\
(5)\qquad &p_{12}p_{21}^2\beta = p_{21}^2 q_{11} - p_{21}q_{21}' + 3q_{21}^2,\\
&p = p_{12}q_{21} - p_{21}q_{12}, \qquad \pi_{12}p_{12} = \pi_{21}p_{21} = 1,\\
&p_{12}r_{12} = q_{12}, \qquad p_{21}r_{21} = q_{21}.
\end{aligned}
$$

The three lines l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$ determine a quadric Q. On each of these lines there will be two points such that tangents at these points to the respective surfaces R_{yz}, $R_{\psi\phi}$, $R_{\eta\theta}$ will be lines of that regulus of Q whose lines intersect l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$. These points are found by factoring the respective quadratic covariants†

$$
(6)\qquad p_{21}y^2 - p_{12}z^2, \qquad \beta\psi^2 - \alpha\phi^2, \qquad \beta\eta^2 + 2p\pi_{12}\pi_{21}\eta\theta - \alpha\theta^2.
$$

In this connection we remark that *the first of these covariants determines the complex points of* l_{yz}.‡

On each of the lines l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$ lie two points at which intersector tangents§ to the surfaces R_{yz}, $R_{\psi\phi}$, $R_{\eta\theta}$ will also be tangents to the respective surfaces $R_{\psi\phi}$, $R_{\eta\theta}$, R_{yz}. These points are given by the respective covariants¶

$$
\begin{aligned}
&(p_{21}\alpha - p_{12}\beta)yz, \quad p(\beta\psi^2 - \alpha\phi^2),\\
&p\pi_{12}\eta^2 + (p_{21}\alpha - p_{12}\beta)\eta\theta - p\pi_{21}\theta^2.
\end{aligned}
\qquad (7)
$$

If the invariant $p_{21}\alpha - p_{12}\beta \not\equiv 0$, *then from the first expression in* (7) *it follows that the intersector tangents to* R_{yz} *at the flecnodes of*

---

* We shall assume that p_{12} and p_{21} do not vanish identically, that is, that neither branch of the flecnode curve of R_{yz} is a straight line.

† T, p. 269 et seq.

‡ A. F. Carpenter, *Ruled surfaces whose flecnode curves have plane branches*, Transactions of this Society, vol. 16 (1915).

§ E. P. Lane, *Ruled surfaces with generators in one-to-one correspondence*, Transactions of this Society, vol. 25 (1923).

¶ T, p. 269 et seq.

l_{yz} are tangent to $R_{\psi\phi}$. If $p_{21}\alpha - p_{12}\beta \equiv 0$, then the intersector tangents to R_{yz} at all points of l_{yz} are tangent to $R_{\psi\phi}$. This conclusion can be verified in the following way.

Let $u = y + \lambda z$ be the expression for the general point on l_{yz} and let us choose for system of reference the tetrahedron $P_yP_zP_\rho P_\sigma$, where the unit point is so chosen that all coordinates of the four vertices are zero except $y_1 = z_2 = \rho_3 = \sigma_4 = 1$, and where

$$\rho = 2y' + p_{12}z, \quad \sigma = 2z' + p_{21}y. \tag{8}$$

Then with the help of equations (4) and (8) the equation of the tangent plane to R_{yz} at P_u is found to be

$$\lambda x_3 - x_4 = 0. \tag{9}$$

This plane cuts $l_{\psi\phi}$ in the point P_v, where

$$v = \psi + k\phi, \quad p_{21}k\lambda = p_{12}.$$

The equation of the tangent plane to $R_{\psi\phi}$ at P_v is, similarly,

$$p_{21}\alpha\lambda x_1 - p_{12}\beta x_2 - (p_{12}^2\beta + 2q_{21}\alpha\lambda)x_3 + (p_{21}^2\alpha\lambda + 2q_{12}\beta)x_4 = 0,$$

and the point P_u lies on this plane for all values of λ, if, and only if, $p_{21}\alpha - p_{12}\beta \equiv 0$.

In a previous paper by the author* it was proved that the osculating planes of the flecnode curve, the complex curve and the harmonic curve of R_{yz} at the points in which these curves cut any line element l_{yz}, will form an axial pencil if, and only if, $p_{21}\alpha - p_{12}\beta = 0$. In view of what precedes we may state the following theorem.

THEOREM 1. *If the planes osculating the flecnode curve, the complex curve, and the harmonic curve of a general ruled surface R at the points of intersection of these curves with each line element of R form an axial pencil, then R and the ruled surface S generated by the axis of this pencil of planes are the focal surfaces of the congruence of intersector tangents of R (or S) with respect to S(or R).*

If the invariant $p \not\equiv 0$, then from (6) *and* (7) *it results that the two points of $l_{\psi\phi}$ at which lines of Q are tangent to $R_{\psi\phi}$ are also the points at which intersector tangents of $R_{\psi\phi}$ are tangent to $R_{\psi\theta}$.* If $p \equiv 0$, then the intersector tangents to $R_{\psi\phi}$ at all points of $l_{\psi\phi}$ are

* A. F. Carpenter, *A theorem on ruled surfaces*, this Bulletin, vol. 34 (1928), pp. 479–481.

tangent to R_{yz}. This can be verified as above. But if $p \equiv 0$, then R_{yz} is identically self-dual.* We can sum up these results in the following statement.

THEOREM 2. *Let the planes osculating the flecnode curve of a general ruled surface R at the two points of intersection of this curve with a line element l of R intersect in s, and let t be the polar reciprocal of s with respect to the linear complex osculating R along l. Then if R is identically self-dual, the two ruled surfaces S and T generated by s and t will be the focal surfaces of the congruence of intersector tangents of S (or T) with respect to T (or S).*

If both $p_{21}\alpha - p_{12}\beta \equiv 0$ and $p \equiv 0$, then the third covariant of (7) vanishes identically. Under these conditions, R_{yz} is a quadric.†

Two of the intersector tangents of R_{yz} at points of l_{yz} must be asymptotic tangents of R_{yz}. We find these points as follows. In order that the curve C_u generated by P_u be an asymptotic curve of R_{yz}, λ must satisfy the differential equation

$$2\lambda' + p_{21}\lambda^2 - p_{12} = 0. \tag{10}$$

By the use of (4) and (10), we find

$$u' = -r_{21}y - \tfrac{1}{2}(p_{21}\lambda^2 + 2r_{12}\lambda - p_{12})z + \pi_{12}\lambda\psi + \pi_{21}\phi. \tag{11}$$

Now the asymptotic tangent at P_u will cut $l_{\psi\phi}$ if $u' + \mu u$ is linear in ψ and ϕ. It follows that

$$u - r_{21} = 0, \quad p_{21}\lambda^2 - 2\mu\lambda + 2r_{12}\lambda - p_{12} = 0,$$

and therefore

$$p_{21}\lambda^2 - 2(r_{21} - r_{12})\lambda - p_{12} = 0. \tag{12}$$

The values of λ from (12) determine the two values of u sought. The corresponding quadratic covariant is found to be

$$p_{21}y^2 + 2(r_{21} - r_{12})yz - p_{12}z^2 = \pi_{21}PQ, \tag{13}$$

where

$$P = p_{21}y + Uz, \; Q = p_{21}y + Vz, \; U = r_{21} - r_{12} + r, \; V = r_{21} - r_{12} - r,$$
$$r = [(r_{21} - r_{12})^2 + p_{12}p_{21}]^{1/2}.$$

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* E. J. Wilczynski, *Projective Differential Geometry*, p. 144; p is a factor of the invariant.

† Wilczynski, loc. cit., p. 150.

The two points in which these asymptotic tangents cut $l_{\psi\phi}$ are given by $\pi_{12}\lambda_1\psi+\pi_{21}\phi$, $\pi_{12}\lambda_2\psi+\pi_{21}\phi$, where λ_1, λ_2 are the roots of (12), and the corresponding covariant is found to be

$$p_{21}\psi^2 - 2(r_{21} - r_{12})\psi\phi - p_{12}\phi^2 = \pi_{12}CD, \tag{14}$$

where $C=p_{21}\psi-V\phi$, $D=p_{21}\psi-U\phi$.

By proceeding as above we find, for the points of l_{yz} at which asymptotic tangents to R_{yz} cut $l_{\eta\theta}$, the covariant

$$p_{21}y^2 - 2(r_{21} - r_{12})yz - p_{12}z^2 = \pi_{21}RS, \tag{15}$$

where $R=p_{21}y-Uz$, $S=p_{21}y-Vz$, and for the points in which these tangents cut $l_{\eta\theta}$ the covariant

$$p_{21}\eta^2 + 2(r_{21} - r_{12})\eta\theta - p_{12}\theta^2 = \pi_{21}AB, \tag{16}$$

where $A=p_{21}\eta+V\theta$, $B=p_{21}\eta+U\theta$.

By comparison of (13), (14), (15), (16) it results that *the line l_{yz} is cut by those faces of the tetrahedron $ABCD$ opposite the vertices A, B, C, D in the respective points P, Q, R, S. It is also seen that P_y, P_z separate harmonically both pairs of points P, R and Q, S.*

Since $r_{21}-r_{12}\equiv 0$ implies $p\equiv 0$, and conversely, it results that if *R_{yz} is identically self-dual the asymptotic tangents to R_{yz} at the complex points of l_{yz} intersect both $R_{\psi\phi}$ and $R_{\eta\theta}$ and are thus rulings of the quadric determined by l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$.*

There exist, in general, for the three lines l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$, two closed intersector sequences of order one.* The pairs of points in which these lines are cut by these sequences are given by the respective covariants

$$\begin{aligned} &p_{21}\alpha\beta y^2 + p(\pi_{12}\alpha + \pi_{21}\beta)yz - p_{12}\alpha\beta z^2, \\ &p_{21}\alpha\beta\psi^2 - p(\pi_{12}\alpha + \pi_{21}\beta)\psi\phi - p_{12}\alpha\beta\phi^2, \\ &p_{12}\beta^2\eta^2 + p(\pi_{12}\alpha + \pi_{21}\beta)\eta\theta - p_{21}\alpha^2\theta^2. \end{aligned} \tag{17}$$

The existence of a closed intersector sequence of order two is assured by the vanishing of the invariant

$$p\pi_{12}\pi_{21}(p_{21}\alpha - p_{12}\beta), \tag{18}$$

and this closure property holds for all such sequences. From (18) it follows that, *under the conditions of either Theorem* 1 *or Theorem* 2, *all intersector sequences of order two are closed.*

* T, pp. 272, 273.

All intersector sequences of order three will be closed if the invariant

$$p^2(p_{21}^2\alpha^2 + p_{12}^2\beta^2) + 2p_{12}^3p_{21}^3\alpha^2\beta^2$$

vanishes. This can happen either if $p=0$, $\alpha=0$ (or $\beta=0$), or $\alpha=0, \beta=0$. In the first case one branch of the flecnode curve of R_{yz} is plane, $(\alpha=0)$, and $R_{\psi\phi}$ degenerates into the tangents of a plane curve. In the second case both branches of the flecnode curve of R_{yz} are plane and $R_{\psi\phi}$ degenerates into a straight line.

It is obvious that in the preceding developments the order of the lines l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$ can be reversed without in any way affecting results. The analysis would be based upon a system of first-order equations of the same type as (4), (5) and obtainable from (4), (5) by simple processes.

THE UNIVERSITY OF WASHINGTON

NOTE ON THE REDUCIBILITY OF ALGEBRAS WITHOUT A FINITE BASE*

BY M. H. INGRAHAM

It is the purpose of this note to discuss the reducibility of linear associative algebras which are not assumed to possess a finite base. J. H. M. Wedderburn,† in seeking to generalize certain theorems on the structure of an algebra, has considered algebras in which restrictions are placed upon the character of the idempotent elements. The summations involved in his study need not be finite. This seems to be one natural line of attack.

J. W. Young‡ has approached the subject from the point of view of the groups involved. His definition of a finite algebra is, however, unsatisfactory, not being sufficiently restrictive.

I have studied infinite algebras in connection with the results that can be obtained by a use of the "axiom of choice" and the theory of transfinite ordinals. This note, however, does not

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† J. H. M. Wedderburn, *Algebras which do not possess a finite base*, Transactions of this Society, vol. 26 (1924), pp. 395–426.

‡ J. W. Young, *A new formulation for general algebras*, Annals of Mathematics, vol. 29 (1927), pp. 47–60. See particularly p. 60.

assume the existence of an infinite base for an algebra but merely the usual postulates for a finite linear associative algebra $A = [a]$ over a field $\Xi = [\xi]$, where the postulate requiring the existence of a finite base is replaced by the following extremely mild postulate: for every pair of elements a_1 and a_2 in the algebra A, we have $1 \cdot a_1 = a_1$ and $a_1 + a_2 \cdot 0 = a_1$, where the elements 1 and 0 are the unit and zero elements respectively of the field Ξ over which the algebra A is taken. There is no difficulty in generalizing the elementary properties of 0 and of a principal unit if it exists.

As in the finite case, a set $S = [s]$ of elements of A is said to be a linear set if for every pair of scalars ξ_1, ξ_2 in Ξ and elements s_1 and s_2 in S, $\xi_1 s_1 + \xi_2 s_2$ is in S. If C is a class of linear sets of elements of A, then the sum of these linear sets is defined as the least linear set containing all the linear sets of C. This is equivalent to defining the sum as the totality of finite linear combinations with scalar coefficients of the elements of the sets of C. The product $S_1 S_2$ of two linear sets S_1 and S_2 is the least linear set containing every element of the form $s_1 s_2$, where s_1 and s_2 are in S_1 and S_2 respectively. I have discussed elsewhere* other properties of linear sets.

As in the case of finite algebras, an invariant proper sub-algebra A_1 of A is one for which both $A_1 A$ and $A A_1$ are contained in A_1. The sum and intersection of a class of invariant sub-algebras is an invariant sub-algebra (or zero) and the sum of two distinct maximal invariant sub-algebras is A. Moreover, we may define the sum of two or more algebras $A_1, A_2, \cdots$ as a direct sum which we denote $A_1(+)A_2(+)\cdots$, or $(\Sigma)A_i$, if for every $i \neq j$, $A_i A_j = A_j A_i = 0$, and the intersection $A_i \wedge A_j = 0$. If A has a principal unit e and $A = A_1(+)A_2$, then A_1 has a principal unit e_1, and A_2 has a principal unit e_2, where $e = e_1 + e_2$. We may prove, as in the finite case, the following theorem.

THEOREM 1. *If A_1 and A_2 are sub-algebras of A either of which has a principal unit, and if $A_1 A_2 = A_2 A_1 = 0$, then $A_1 \wedge A_2 = 0$ and $A_1(+)A_2$ is a direct sum.*

THEOREM 2. *If A has an invariant sub-algebra A_1 which possesses a principal unit e_1, then A is reducible and has A_1 for one component.*

* M. H. Ingraham, *A general theory of linear sets*, Transactions of this Society, vol. 27 (1925), pp. 163–196.

J. H. M. Wedderburn, in his above mentioned paper, proves this theorem on the basis of a postulate which requires that for every linear set there exists a supplementary set such that the sum of the two is A and the intersection zero. His proof, however, makes no essential use of this hypothesis. The first real difference in this theory from that of finite algebras comes in attempting to generalize the following theorem.*

Any reducible finite algebra A with a principal unit e can be expressed as the direct sum of irreducible algebras each with a principal unit, in one way and only one way apart from the arrangement of the component algebras.

This theorem is not true in the extended theory for there exist algebras of infinite order with principal units which are not expressible as the direct sum of irreducible algebras. Although the existence of a complete reduction of a reducible algebra can not be proved in general, it will be seen as an immediate corollary of Theorem 4 that the uniqueness, except for order of components of any complete reduction into sub-algebras each with a principal unit, does hold in general. This situation is given by Theorems 3 and 4.

THEOREM 3. *If A can be expressed as the direct, sum of sub-algebras in two ways $A=(\Sigma)A_i=(\Sigma)B_i$, such that $AA_i=A_iA=A_{ij}$ and $B_iA=AB_i=B_i$, then each A_i and B_i can be expressed as the direct sum of one or more sub-algebras $A_i=(\Sigma)A_{ij}$, $B_i=(\Sigma)B_{ij}$ such that apart from order the A_{ij}'s and the B_{ij}'s are identical.*

THEOREM 4. *If A has a principal unit, it can not be expressed as the direct sum of more than a finite number of sub-algebras.*

PROOF OF THEOREM 3. Let $A_{ij}=B_{ji}$ equal the intersections of A_i and B_j.

$A_i=A_iA=\Sigma_jA_iB_j$. But A_iB_j is a portion of the intersection of A_i and B_j, and hence $A_i=\Sigma_jA_{ij}$ and, moreover, it is a direct sum since A_{ij} is in B_j and for every $j\neq k$, $B_jB_k=B_kB_j=0$ and $(B_j\wedge B_k)=0$. Similarly, $B_j=(\Sigma)_iB_{ji}$. Hence $A=(\Sigma)_iA_i=(\Sigma)_{ij}A_i=(\Sigma)_j(B_j)=(\Sigma_{ij}B_{ji})$, and apart from order, the A_{ij} and the B_{ji} are identical. Moreover, it should be noted that $A_{ij}=A_iB_j=B_jA_i$.

* Dickson, *Algebras and their Arithmetics*, p. 35.

COROLLARY. *If an algebra A can be reduced to the direct sum of irreducible algebras A_i, each A_i having the property that $A_iA = AA_i = A_i$, this reduction is, apart from order, unique.*

It is obvious that $AA_i = A_iA = A_i$ if A_i contains a principal unit. Hence reduction of an algebra to irreducible subalgebras each with a principal unit is unique.

In order to prove Theorem 4, consider $A = (\Sigma)A_i$ and containing a principal unit e. The principal unit e can be expressed uniquely as a sum of a finite number of elements not more than one of which belongs to any A_i. Then $e = \sum a_i$, where, except for a finite number of values of i, $a_i = 0$ if A_j is such that $a_j \neq 0$; then $A_j = eA_j = 0$, and the theorem follows at once.

The following three examples somewhat clarify the situation. Example 1 is an algebra which, although it contains no principal unit, is the direct sum of an infinite number of irreducible sub-algebras each with a principal unit and hence, by the corollary to Theorem 4, is uniquely reducible. Example 2 is an algebra with a principal unit containing Example 1 and its invariant sub-algebras as invariant sub-algebras, and hence by Theorem 4 is not reducible to the direct sum of irreducible sub-algebras. Example 3 is the most interesting of the group, being an algebra such that every invariant sub-algebra contains a proper invariant sub-algebra with a principal unit, and hence is reducible. All of these algebras contain denumerable finitely linearly independent bases and hence we do not believe that any simple postulates as to bases for the infinite algebras will yield appreciably greater results in this portion of the theory. Although all the examples are commutative, non-commutative examples can easily be built up as the direct product of these and finite non-commutative division algebras.

EXAMPLE 1. Let $A =$ [all sequences $\{\xi_i\}$ of scalars which are zero except for a finite number of elements]. Let addition, multiplication be the addition and multiplication of corresponding elements, and scalar multiplication the multiplication of each element by the given scalar. Let A_i be all the sequences for which $\xi_j = 0$, $(i \neq j)$. Then $A = (\Sigma)A_i$.

EXAMPLE 2. Let $A =$ [all sequences $\{\xi_i\}$ of scalars such that $\xi_i = \xi_{i+1}$ except for a finite set of values for i]. Let addition, multiplication, and scalar multiplication be defined as above.

Each A_i of Example 1 is an invariant sub-algebra of A but $A \neq (\Sigma)A_i$.

EXAMPLE 3. Consider $u_0 = u_1 + v_1$, where

$$u_0^2 = u_0, \qquad u_1^2 = u_1, \qquad v_1^2 = v_1,$$

$$u_0u_1 = u_1u_0 = u_1, \qquad u_1v_1 = 0, \qquad u_0v_1 = v_1u_0 = v_1.$$

In a recursive manner, let

$$u_1 = u_2 + v_2, \qquad v_1 = u_3 + v_3,$$

where the table of multiplication for u_1, u_2, v_2 can be gotten from that for u_0, u_1 and v_1 by replacing u_0 by u_1, u_1 by u_2, v_1 by v_2, and the multiplication table for v_1, u_3 and v_3 can be gotten in a similar manner by replacing u_0 by v_1, u_1 by u_3, and v_1 by v_3. In a similar manner, express each of u_2, v_2, u_3 and v_3 as sums of a pair of elements, etc. If we call u_1 and v_1 direct descendents of u_0, u_2 and v_2 direct descendents of u_1 and descendents of u_0, etc., the table of multiplication can be stated in the following geneological fashion. Each u or v is idempotent. An element times its descendent is the descendent; an element times any brother or cousin no matter how far removed is zero. It is also clear that the u's form a base for the total algebra. Moreover, if any v or u is in an invariant sub-algebra A_1 of A, it and all its descendents form a base for an invariant sub-algebra A_2 of A_1 and A of which it is the principal unit. Suppose then A_1 contains an element $a_1 = \sum_{i=1}^{n} \xi_i u_i$. Let u_n be a u in the lowest generation represented in a_1. Then $a_1u_n = (\sum_1^n \xi_i)u_n$. If $\sum_1^n \xi_i \neq 0$, A_1 contains u_n. If $\sum_1^n \xi_i = 0$ and if w is the direct ancestor of u_n, then $a_1w = \sum_1^{n-1} \xi_i w + \xi_n u_n = -\xi_n v_n$ and A_1 contains v_n, hence A_1 contains either u_n or v_n, and hence each of their descendents and hence a proper invariant sub-algebra with a principal unit.

THE UNIVERSITY OF WISCONSIN

HEXAGONAL SYSTEMS OF SEVEN LINES IN A PLANE*

BY LOUISE D. CUMMINGS

1. *Introduction.* This paper concerns the determination of the non-equivalent systems of seven real lines in a plane when no three of the lines are co-punctual, and the investigation is limited to that subdivision of the problem where, in a system of seven lines, some six form a convex hexagon. It is shown that exactly eight non-equivalent arrangements exist, and incidentally the integers showing the numbers of polygons of 3, 4, 5, 6, 7 sides which occur in each of the eight systems are tabulated. The method, developed by Professor H. S. White, of the unique characterization of a line by means of the contiguous line-segments in a system, has been used to determine a "mark" for every line in the eight systems. The seven marks of a system are employed to prove the non-equivalence of systems and to determine the substitution connecting two equivalent systems.

2. *Basic Hexagon.* Six lines in a plane, no three in any point, form 30 segments, and for the hexagonal subdivision here considered, divide the plane into one hexagon, six triangles, and nine quadrilaterals. For easier visualization, an irregular hexagon with sides produced indefinitely and with all 15 intersections in the finite plane is considered. The 30 segments of the six lines may be assigned to three classes, namely, 6 primary, separating the hexagon from the triangles, 12 secondary, separating the triangles from quadrilaterals, 12 tertiary, separating, each, two quadrilaterals. The nine quadrilaterals are separable into two types: (1) six quadrilaterals with sides two adjacent secondary segments and two adjacent tertiary segments, (2) three quadrilaterals with all sides tertiary segments. The 30 segments separate into five continuous broken lines as follows: the 6 primary segments bound the hexagon, the 12 secondary segments surround the six triangles, and the 12 tertiary segments separate into the three boundaries of the three quadrilaterals of the second type.

3. *Notation and Method.* When, in a set of seven lines, any six form a hexagon, that hexagon is utilized as unique initial figure.

* Presented to the Society, October 31, 1931.

The six vertices in order on the hexagon are marked 1, 2, 3, 4, 5, 6. On the broken line of the secondary segments insert the letters a, b, c, d, e, f, in order, a being the vertex of the triangle whose base is the primary segment 12. Mark the three intersections of the three pairs of opposite sides of the hexagon L, M, N, in the same cyclic order, L being the intersection of the pair of opposite sides 12 and 45. Mark the six intersections of the seventh line or secant as p, q, r, s, t, u. Name the six lines forming the hexagon $l_1, l_2, l_3, l_4, l_5, l_6$ and the seventh line l_7. Two subdivisions of the problem are considered: Case I and Case II.

CASE I. *The seventh line crosses the hexagon*, intersecting two primary segments, two secondary segments, and two tertiary segments. In the hexagon the two primary segments may be (i) adjacent sides, (ii) alternate sides, (iii) opposite sides.

CASE II. *The seventh line lies outside the hexagon* and intersects (i) six tertiary segments, (ii) four tertiary and two secondary segments, (iii) two tertiary and four secondary segments.

CASE I (i). The seventh line enters the hexagon across a triangle, for example the triangle $a12$, and cutting the segments $a1$, 12, emerges from the hexagon across the consecutive side 23 and must cut the segments $3b$, cL, fM. In the pentagonal subdivision of seven real lines in a plane, H. S. White has employed a unique mark for any secant in a system of lines by means of the contiguous line-segments which that secant crosses. Two segments are *contiguous* if they have a common extremity, and in determining the mark of a secant contiguous segments are written so as to form a continuous broken line. For example the line l_7 crosses the contiguous segments $a1$, 12, 23, $3b$ and two separate segments cL, fM, and the mark for the line l_7 is written $l_7(a123b \cdot cL \cdot fM)$ or for brevity $l_7(4 \cdot 1 \cdot 1)$. The line l_7 intersecting two consecutive sides of the basic hexagon divides the hexagon into a triangle and a heptagon. In this heptagon the seventh line and the initial six lines now occupy similar roles, each being a side of the heptagon, hence we know, without further examination, that all seven lines have the same mark $(4 \cdot 1 \cdot 1)$. This system designated as system (1) is identified by the *characteristic* $(4 \cdot 1 \cdot 1)^7$.

CASE I (ii).* The seventh line or secant enters the hexagon across the segments $a1$, 12 but emerges across the side of the hexagon 34, alternate to the side 12, and entering the triangle $c34$ may leave the triangle across $3c$ or $4c$. The path across $3c$ is unique and the secant cuts the six segments $a1$, 12, 43, $3c$, cL, fM and the mark of this line $l_7(a12 \cdot 43cL \cdot fM)$ is $l_7(3 \cdot 2 \cdot 1)$. This investigation is concerned primarily with the development of a necessary and sufficient test for the equivalence or non-

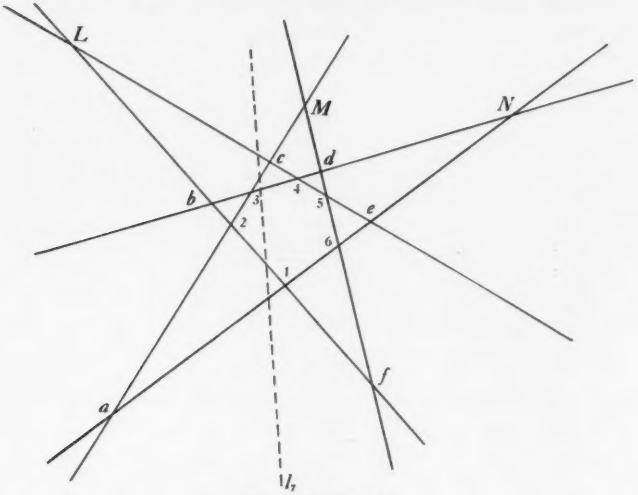

FIG. 1. System (2)

equivalence of two systems of seven real lines in a plane, where two systems are equivalent if a one-to-one relation exists between the lines and polygons of the two systems. The line $l_7(3 \cdot 2 \cdot 1)$ of system (2) cannot be transformed into any line with the mark $(4 \cdot 1 \cdot 1)$ and therefore system (2) is not equivalent to system (1). However, for purposes of comparison with other systems, it is necessary in some cases to have all seven marks of each system and the marks of system (2) consist of four of this new kind $(3 \cdot 2 \cdot 1)$ and three of the first kind $(4 \cdot 1 \cdot 1)$, and system (2) has the characteristic $(4 \cdot 1 \cdot 1)^3 + (3 \cdot 2 \cdot 1)^4$. If the line

* Figure 1.

l_7 crosses the segments $a1$, 12, 34, $4c$ into the quadrilateral $4cMd$, two paths of exit are possible, either across the side cM adjacent to $4c$ or across the segment dM opposite to $4c$, and the two paths must be examined. On the first path the six segments cut by the secant are $a1$, 12, 34, $4c$, cM, Mf and the mark of the secant is $l_7(a12\cdot 34cMf)=(4, 2)$. This mark does not occur in system (1) nor in system (2) and belongs to a new system (3) with the characteristic $(5\cdot 1)+(4\cdot 2)^2+(4\cdot 1\cdot 1)+(3\cdot 2\cdot 1)^3$. Examination of the second possible path across $4c$ gives a system (3′) with the same characteristic as that derived for system (3). A comparison now of the seven marks of these two systems determines very easily the substitution S which transforms system (3′) into system (3), namely, $S\equiv(14)(23)(56)(ac)(b)(df)(e)(LN)(M)(pq)(ru)(st)$.

CASE I (iii). The seventh line crosses that hexagon on two opposite sides 12 and 45, and the investigation of all possible paths shows the existence of three new non-equivalent systems (4), (5) and (6) tabulated below in Table I.

CASE II (i). The secant crosses six tertiary segments giving rise to two non-equivalent systems designated in the following table as (7) and (8). In system (7) the three Pascalian points L, M, N all lie on the same side of the seventh line, and the seventh line crosses the three quadrilaterals of the second type and three *consecutive* quadrilaterals of the first type. In system (8) the points L, M, N do not all lie on the same side of the seventh line, and the seventh line crosses the three quadrilaterals of the second type and three *alternate* quadrilaterals of the first type.

CASE II (ii). The seventh line crosses two secondary segments and then may cross four tertiary segments over two different paths, giving rise to two systems (3_1) and (3_2) each of which is equivalent to system (3). This result was to be expected since in system (3) a second hexagon, including the line l_7, exists, which may be used as basic hexagon. If the segments are renamed with respect to this second hexagon the secant line, now l_2, crosses four tertiary and two secondary segments.

CASE II (iii). The path of the seventh line across four secondary segments and two tertiary segments is unique and gives rise to one system (2_1) which is equivalent to system (2), a result in

agreement with the fact that system (2) contains also a second hexagon.

4. *Table of Non-Equivalent Hexagonal Systems.* In the following Table I, the marks for each of the eight non-equivalent hexagonal systems are given and also, incidentally, the actual polygonal division of the projective plane, satisfying of course the Euler equation

$$s_7 + s_6 + s_5 + s_4 + s_3 = \frac{n^2 - n + 2}{2},$$

where s_i, the number of polygons of i sides, is tabulated for each of the eight systems. The three non-equivalent systems, derived by H. S. White in the pentagonal subdivision of his paper,* have been adjoined in Table I for convenience in comparison, and are designated here as (9), (10), (11).

TABLE I

SYSTEM	CHARACTERISTICS							POLYGONS				
	(6·0)	(5·1)	(4·2)	(4·1·1)	(3·3)	(3·2·1)	(2·2·2)	s_7	s_6	s_5	s_4	s_3
(1)				7				1			14	7
(2)				3		4			1	1	13	7
(3)		1	2	1		3			1	2	11	8
(4)				2		4	1			3	12	7
(5)	1	2			2	2				5	8	9
(6)		2			1	4				4	10	8
(7)	5		2						1	5	5	11
(8)			6				1		1	3	9	9
	Pentagonal Systems											
(9)						6	1			3	12	7
(10)			2		1	2	2			4	10	8
(11)	3				3		1			6	6	10

5. *Properties shown by Table* I. The numbers listed for the polygonal divisions of the plane for these eight hexagonal systems agree with the result of this investigation, that the eight systems are non-equivalent, and the same fact is true for the table of the pentagonal systems. However, a comparison of the two parts of the table shows that systems (6) and (10) have the same polygonal numbers 4, 10, 8, and hence that equality of polygonal numbers is a necessary but not a sufficient condition for the equivalence of two systems. The characteristics of these two systems, however, show immediately that the two systems are non-equivalent. The non-equivalent systems (4) and (10) also have equality of polygonal numbers.

* *The plane figure of seven real lines*, this Bulletin, vol. 38 (1932), pp. 59–65.

6. *Conclusion.* This method of analysis of straight-line nets by the contiguous segments, as herein extended to all the lines in the system, regardless of their relation to a pentagon, hexagon, or other basal polygon, is applicable to any number n of straight lines and is even not restricted to the case that only $m=2$ lines shall pass through a point. The method furnishes a *necessary and sufficient* test for the equivalence or the non-equivalence of two systems of straight lines, and in the case of two equivalent systems this method simplifies the discovery of the substitution which transforms the one system into the other system.

VASSAR COLLEGE

A PRACTICAL METHOD FOR THE MODULAR REPRESENTATION OF FINITE OPERATIONS AND RELATIONS*

BY B. A. BERNSTEIN AND NEMO DEBELY

1. *Introduction.* In previous papers one of the writers developed a general theory for the concrete representation of arbitrary operations and relations in a finite class of elements.† Let p be a prime, and let a mod p denote the least positive integer obtained from integer a by dropping multiples of p. Consider the function $f(x)$ given by

$$f(x) = c_0 + c_1x + \cdots + c_{p-1}x^{p-1}, \qquad \text{mod } p, \tag{1}$$

where x ranges over the complete system of p-residues $0, 1, \cdots, p-1$, and where the coefficients c_i are among the p-residues. The general theory is based on the fact that any unary operation in a class K of p elements, the operation satisfying the condition of closure, can be represented by a polynomial of form (1). But when the number of elements in K is large, the calculation of (1) by the method of the general theory is very laborious, for the work involves, for a class of p elements, the computation modulo p of p determinants each of order $p-1$. For an m-ary operation or an m-adic relation where $m>2$, the calculation of the representation by the method of the general theory is very laborious

* Presented to the Society, April 5, 1930.

† See the Proceedings of the International Mathematical Congress, Toronto, 1924, p. 207, and this Bulletin, vol. 32, p. 533.

even when p is as small as 3. Moreover, the method of the theory is not at all adapted to cases in which p is a letter instead of a given number. The present paper gives a method of obtaining with extreme ease the representations of the theory for operations and relations of any complexity and for p literal or a number of any magnitude.

2. *The Unit-Zero Functions.* The method of the present paper makes fundamental the notion of *unit-zero function.* A function $f(x)$ will be called a *unit-zero function with respect to a* if $f(x)=1$ or 0, according as $x=a$ or $x\neq a$. In general, a function $f(x_1, x_2, \cdots, x_m)$ will be called a *unit-zero function with respect to the sequence* $a_1, a_2, \cdots, a_m$ if $f(x_1, x_2, \cdots, x_m)=1$ or 0, according as the equalities $x_i=a_i$, $(i=1, 2, \cdots, m)$, do or do not all hold. The functions with which our theory is concerned are all *polynomials modulo p*, where p is prime. A unit-zero function with respect to the sequence $a_1, a_2, \cdots, a_m$, if the function be a polynomial modulo p in $x_1, x_2, \cdots, x_m$, will be denoted by $(x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m)_p$.

The unit-zero functions $(x; a)_p$ and $(x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m)_p$ can be readily written down. Indeed, by Fermat's theorem, we have

$$(2) \qquad (x; a)_p = 1 - (x-a)^{p-1}, \bmod p,$$

$$(2') \qquad = 1 + (p-1)\sum_{k=0}^{p-1} a^k x^{p-1-k}, \bmod p.$$

And, evidently,

$$(3) \quad (x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m)_p = (x_1; a_1)_p (x_2; a_2)_p \cdots (x_m; a_m)_p.$$

From the nature of the unit-zero function we have

$$(4) \quad a(x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m)_p + b(x_1, x_2, \cdots, x_m; b_1, b_2, \cdots, b_m)_p = a, \text{ or } b, \text{ or } 0,$$

according as $x_i=a_i$ all hold, or $x_i=b_i$ all hold, or neither $x_i=a_i$ all hold nor $x_i=b_i$ all hold, $(i=1, 2, \cdots, m)$.

3. *Representations.* The method of representing finite operations and relations by means of unit-zero functions can now be stated. Let K be a finite class of n elements. These elements may be denoted by $0, 1, \cdots, n-1$. The representations of

operations O and relations R in K are given in cases (A)-(E) below.

(A). *O a K-closing m-ary operation, n a prime p.* There is a K-element $e_{a_1 a_2 \cdots a_m}$ for every sequence of m elements $a_1, a_2, \cdots, a_m$ of K. From (4), the representation of O is the function

$$(5) \quad \sum_{a_1=0}^{p-1} \cdots \sum_{a_{m-1}=0}^{p-1} \sum_{a_m=0}^{p-1} e_{a_1 a_2 \cdots a_m}(x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m)_p.$$

Thus, the representation of the operation

(i)

	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

is the function $f(x, y)$ given by

$$\text{(ii)} \quad \begin{aligned} f(x, y) &= 2(x, y; 0, 1)_3 + (x, y; 0, 2)_3 + (x, y; 1, 0)_3 \\ &\quad + 2(x, y; 1, 2)_3 + 2(x, y; 2, 0)_3 + (x, y; 2, 1)_3 \\ &= x + 2y, \bmod 3. \end{aligned}$$

(B). *O an m-ary operation not K-closing, n a prime p.* There are sequences

$$\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1m}; \alpha_{21}, \alpha_{22}, \cdots, \alpha_{2m}; \cdots; \alpha_{k1}, \alpha_{k2}, \cdots, \alpha_{km}$$

to which no K-elements correspond. Let O' be the operation obtained from O by assigning some K-element, 0 for convenience, to each of these sequences. Let $\phi(x_1, x_2, \cdots, x_m)$ be the function, obtained as in (A), representing O'. The representation of O is the function

$$(6) \quad \phi(x_1, x_2, \cdots, x_m) + \sum_{i=1}^{k} 0/\{1 - (x_1, x_2, \cdots, x_m; \alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{im})_p\}.$$

Thus, consider the operation $f(x)$ defined by

(iii)

x	0	1	2	3	4
$f(x)$	2	—	0	—	1

where $f(1)$ and $f(3)$ do not belong to the class $0, 1, \cdots, 4$. Take the operation $\phi(x)$ defined by

(iv)

x	0	1	2	3	4
$\phi(x)$	2	0	0	0	1

The representation of (iv) is given by

(v) $$\phi(x) = 2(x; 0)_5 + (x; 4)_5.$$

Hence, the representation of (iii) is given by

(vi) $$f(x) = \phi(x) + 0/\{1 - (x; 1)_5\} + 0/\{1 - (x; 3)_5\}$$
$$= 3 - 2x^4 - (x-4)^4 + 0/(x-1)^4 + 0/(x-3)^4, \text{ mod } 5.$$

(C). *O an m-ary operation in K, n not prime.* Consider a class K' of p elements $0, 1, \cdots, n-1, \cdots, p-1$, where *$p$ is a prime exceeding n.* Let O' be any operation in K' identical with O for all the sequences of m elements taken from the K-elements $0, 1, \cdots, n-1$. For convenience, we may have O' assign 0 to each of the sequences in K' that are not in K. Let $\phi(x_1, x_2, \cdots, x_m)$ be the representation of O', obtained as in (A) or (B). The representation of O is the function $\phi(x_1, x_2, \cdots, x_m)$ *in which the x_i range over the K-elements* $0, 1, \cdots, n-1$.

Thus, the representation of the operation $f(x)$ defined by

(vii)

x	0	1	2	3
$f(x)$	2	1	0	4

is the representation of the operation $\phi(x)$ defined by

(viii)

x	0	1	2	3	4
$\phi(x)$	2	1	0	4	0

with x ranging over the elements 0, 1, 2, 3.

(D). *R an m-adic relation in K, n a prime p.* Let the sequences which satisfy R be

$$\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1m}; \alpha_{21}, \alpha_{22}, \cdots, \alpha_{2m}; \cdots; \alpha_{k1}, \alpha_{k2}, \cdots, \alpha_{km}.$$

The representation of R is the equation modulo p

(7) $$\sum_{i=1}^{k}(x_1, x_2, \cdots, x_m; \alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{im})_p = 1.$$

Thus, the representation of the dyadic relation defined by

(ix)

	0	1	2
0	−	+	+
1	−	−	+
2	−	−	−

is the modular equation

(x) $$(x, y; 0, 1)_3 + (x, y; 0, 2)_3 + (x, y; 1, 2)_3 = 1.$$

(E). *R an m-adic relation in K, n not prime.* Consider a class K' of p elements $0, 1, \cdots, n-1, \cdots, p-1$, where p is a prime exceeding n. Let R' be any m-adic relation in K' identical with R for all the sequences of m elements taken from the K-elements $0, 1, \cdots, n-1$. For convenience we may have R' contradicted for all the sequences in K' that are not in K. Let the representation of R', obtained as in (D), be the equation modulo p

(8) $$\phi(x_1, x_2, \cdots, x_m) = 1.$$

The representation of R is equation (8), with the x_i *ranging over the K-elements* $0, 1, \cdots, n-1$.

Thus, the representation of the relation R defined by

(xi)

	0	1	2	3
0	−	+	+	+
1	−	−	+	+
2	−	−	−	+
3	−	−	−	−

is the representation of the relation R defined by

(xii)

	0	1	2	3	4
0	−	+	+	+	−
1	−	−	+	+	−
2	−	−	−	+	−
3	−	−	−	−	−
4	−	−	−	−	−,

with x, y ranging over the elements 0, 1, 2, 3.

THE UNIVERSITY OF CALIFORNIA

THE USE OF FRACTIONAL INTEGRATION AND DIFFERENTIATION FOR OBTAINING CERTAIN EXPANSIONS IN TERMS OF BESSEL FUNCTIONS OR OF SINES AND COSINES

BY W. O. PENNELL

1. *Introduction.* Under certain conditions fractional integration or differentiation of a sine function will lead to a Bessel function and vice versa. Likewise the fractional integration or differentiation of a cosine function will lead to Struve's function. It follows, when the process is legitimate, that a known expansion in sines can be converted to an expansion in Bessel functions, or a known expansion in Bessel functions can be converted to a sine expansion, by the simple expedient of term-by-term fractional integration or differentiation.

2. *Fractional Integration and Differentiation of Sine and Cosine.* Fractional integration* with the Heaviside operator p is given by the equation

$$p^{-\nu}f(x) = \int_0^x \frac{(x-\lambda)^{\nu-1}}{\Gamma(\nu)} f(\lambda)d\lambda, \tag{1}$$

where $\nu>0$. Fractional differentiation* is given by

$$p^{\nu}f(x) = \frac{d^b}{dx^b}\int_0^x \frac{(x-\lambda)^{c-1}}{\Gamma(c)} f(\lambda)d\lambda, \tag{2}$$

where $\nu>0$, $0<c<1$, b is a positive integer, and $\nu=b-c$. If (1) is applied to $f(x)=x^n$ the result is

$$p^{-\nu}x^n = \frac{\Gamma(n+1)x^{n+\nu}}{\Gamma(n+\nu+1)}, \quad (\nu>0,\ n>-1). \tag{3}$$

If (2) is applied to x^n,

$$p^{\nu}x^n = \frac{\Gamma(n+1)x^{n-\nu}}{\Gamma(n-\nu+1)}, \quad (\nu>0,\ n>-1). \tag{4}$$

* For bibliography on fractional integration see H. T. Davis, *The application of fractional operators to functional equations*, American Journal of Mathematics, vol. 49 (1927), pp. 123–142.

If $n-\nu+1$ is zero or a negative integer, the right hand side of (4) is to be interpreted as the limit which that expression approaches as ν approaches the value which causes $n-\nu+1$ to become zero or a negative integer.

It follows from (3) and (4) that $p^{\nu}x^{n}=\Gamma(n+1)x^{n-\nu}/\Gamma(n-\nu+1)$ for all values* of ν, for $n>-1$.

By applying process (3) or (4) term by term to the power series representing the sine or cosine we get

$$(5)\quad p^{-\nu+1/2}\sin ax^{1/2}=\left(\frac{2}{a}\right)^{\nu-1}\pi^{1/2}x^{\nu/2}J_{\nu}(ax^{1/2}),$$

$$(6)\quad p^{-\nu+1/2}\cos ax^{1/2}=\frac{x^{\nu-1/2}}{\Gamma(\nu+\frac{1}{2})}-\left(\frac{2}{a}\right)^{\nu-1}x^{\nu/2}\pi^{1/2}H_{\nu}(ax^{1/2}),$$

where H_{ν} denotes Struve's function of order ν. That the term-by-term fractional integration and differentiation is legitimate may be shown as follows. Since the power series for $\sin ax^{1/2}$ is uniformly convergent for all finite values of x it may be written as

$$(7)\qquad \sin ax^{1/2}=\sum_{n=1,2\cdots}^{n}(-1)^{n-1}\frac{(ax^{1/2})^{2n-1}}{(2n-1)!}+R_{n},$$

where $|R_{n}|<\epsilon$, ϵ being chosen arbitrarily small and positive, and corresponding to it is a positive integer n, independent of x. By applying process (1), fractional integration, to both sides of (7), we obtain

$$p^{-\nu+1/2}\sin ax^{1/2}=\sum_{n=1,2\cdots}^{n}(-1)^{n-1}p^{-\nu+1/2}\frac{(ax^{1/2})^{2n-1}}{(2n-1)!}+p^{-\nu+1/2}R_{n}.$$

But

$$|p^{-\nu+1/2}R_{n}|<\left|\frac{\epsilon x^{\nu-1/2}}{\Gamma(\nu+\frac{1}{2})}\right|,$$

and, since $\nu\geqq+1/2$, this term can be made as small as we like, for any finite value of x.

Fractional differentiation, shown by (2), is fractional integration followed by ordinary differentiation. Since the fractional integral of $\sin ax^{1/2}$ or $\cos ax^{1/2}$ leads to a Bessel function of the first kind or Struve's function, and since the power series repre-

* Including $\nu=0$, as p^{0} will be defined as the identical operation.

senting these functions may be differentiated term by term, it follows that the process leading to (5) and (6) is legitimate for all real values of ν, positive or negative.

By transposing the operator in (5) and (6) we get

$$p^{\nu-1/2}\left(\frac{2}{a}\right)^{\nu-1}\pi^{1/2}x^{\nu/2}J_\nu(ax^{1/2}) = \sin ax^{1/2}, \tag{8}$$

$$p^{\nu-1/2}\left(\frac{2}{a}\right)^{\nu-1}\pi^{1/2}x^{\nu/2}H_\nu(ax^{1/2}) = 1 - \cos ax^{1/2}. \tag{9}$$

3. *Fractional Integration of Fourier-Bessel Series.* This series for $f(x)$ is given by

$$f(x) = \sum_{m=1,2\cdots}^{\infty} a_m J_\nu(j_m x), \quad (0 < x < 1, \nu \geqq -\tfrac{1}{2}), \tag{10}$$

where

$$a_m = \frac{2}{J_{\nu+1}^2(j_m)}\int_0^1 \lambda f(\lambda)J_\nu(j_m\lambda)d\lambda, \tag{11}$$

and where j_m is the mth positive root of $J_\nu(x)=0$. From (10),

$$x^{\nu/2}f(x^{1/2}) = \sum_{m=1,2\cdots}^{\infty} a_m x^{\nu/2}J_\nu(j_m x^{1/2}). \tag{12}$$

It can be shown* that (12), with certain restrictions on $f(x)$, is uniformly convergent in the interval $0<x<1$, for $\nu\geqq 0$. The series may, therefore, be fractionally integrated term by term. We get by applying process (8) to (12)

$$p^{\nu-1/2}x^{\nu/2}f(x^{1/2}) = \frac{1}{\pi^{1/2}}\sum_{m=1,2\cdots}^{\infty} a_m\left(\frac{2}{j_m}\right)^{1-\nu}\sin(j_m x^{1/2}), \quad (\tfrac{1}{2}\geqq\nu\geqq 0).$$

If we put

$$\phi(x^{1/2}) = p^{\nu-1/2}x^{\nu/2}f(x^{1/2}),$$

then

$$f(x^{1/2}) = x^{-\nu/2}p^{-\nu+1/2}\phi(x^{1/2}), \tag{13}$$

and

$$\phi(x) = \frac{1}{\pi^{1/2}}\sum_{m=1,2\cdots}^{\infty} a_m\left(\frac{2}{j_m}\right)^{1-\nu}\sin(j_m x), \tag{14}$$

$$(\tfrac{1}{2}\geqq\nu\geqq 0,\ 0<x<1),$$

* See Watson, *Theory of Bessel Functions*, pp. 594–615. Also C. N. Moore, *On the uniform convergence of the developments in Bessel functions*, Transactions of this Society, vol. 12 (1911), pp. 181–206.

where a_m is given by (11). The function $f(x)$ is subject to the restrictions sufficient for a function to be expansible by a Fourier-Bessel series. The function $\phi(x)$ is given by (13). If $f(x) = \sum a_n x^n$, a power series, (13) shows that

$$\psi(x) = \sum_n a_n \frac{\Gamma(n/2 + \nu/2 + 1)}{\Gamma(n/2 - \nu/2 + 3/2)} x^{n-\nu+1}.$$

We have here the expansion of a function in a series of sine terms, and the expansion is not periodic. As far as the author is aware the expansion (14) has never before been published.

EXAMPLE. Let us start with the known expansion

$$x^\nu = \sum_{m=1,2\cdots}^{\infty} \frac{2J_\nu(j_m x)}{j_m J_{\nu+1}(j_m)}, \qquad (\nu \geqq -\tfrac{1}{2},\ 0 \leqq x < 1). \tag{15}$$

Substituting $x^{1/2}$ for x, multiplying both sides by $x^{\nu/2}$, operating on both sides by $p^{\nu-1/2}$, and finally replacing $x^{1/2}$ by x, we may write (15) in the form

$$x = \sum_{m=1,2\cdots}^{\infty} \frac{1}{2^{\nu-1}(j_m)^{2-\nu} J_{\nu+1}(j_m)\Gamma(\nu+1)} \sin j_m x, \tag{16}$$

where $0 \leqq x < 1$, $0 \leqq \nu \leqq 1/2$, and j_m is the mth positive root of $J_\nu(x) = 0$.

4. *Fractional Integration of Fourier Series.* Let us consider the Fourier series

$$f(x) = \tfrac{1}{2}b_0 + \sum_{n=1,2\cdots}^{\infty} \left[a_n \sin\frac{n\pi x}{c} + b_n \cos\frac{n\pi x}{c} \right], \tag{17}$$

where

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x) \sin\frac{n\pi x}{c}\,dx, \quad b_n = \frac{1}{c}\int_{-c}^{c} f(x) \cos\frac{n\pi x}{c}\,dx.$$

In (17), replacing x by $x^{1/2}$ and operating on both sides by $p^{-\nu+1/2}$, we find

$$\begin{aligned} p^{-\nu+1/2} f(x^{1/2}) &= \tfrac{1}{2}b_0 \frac{x^{\nu-1/2}}{\Gamma(\nu+\tfrac{1}{2})} + \sum_{n=1,2\cdots}^{\infty} \pi^{1/2}\left(\frac{2c}{n\pi}\right)^{\nu-1} x^{\nu/2} \\ &\cdot \left[a_n J_\nu\left(\frac{n\pi x^{1/2}}{c}\right) - b_n H_\nu\left(\frac{n\pi x^{1/2}}{c}\right)\right] + \sum_{n=1,2\cdots}^{\infty} b_n \frac{x^{\nu-1/2}}{\Gamma(\nu+\tfrac{1}{2})}, \end{aligned} \tag{18}$$

$$\begin{aligned} x^{-\nu/2} p^{-\nu+1/2} f(x^{1/2}) &= \frac{f(0)\, x^{\nu/2-1/2}}{\Gamma(\nu+\tfrac{1}{2})} + \pi^{1/2}\left(\frac{2c}{\pi}\right)^{\nu-1} \sum_{n=1,2\cdots}^{\infty} \frac{1}{n^{\nu-1}} \\ &\cdot \left[a_n J_\nu\left(\frac{n\pi x^{1/2}}{c}\right) - b_n H_\nu\left(\frac{n\pi x^{1/2}}{c}\right)\right], \end{aligned}$$

where $f(0)=\frac{1}{2}[f(+0)+f(-0)]$. Since the integral of a function defined and satisfying Dirichlet's conditions in a given interval is equal to the term-by-term integral of the corresponding Fourier series, it follows that the fractional integral of the function will equal the term-by-term fractional integral of the Fourier series. This is so because fractional integration of order $\nu-1/2$ (see equation (1)) is equivalent to multiplying each term by $(x-\lambda)^{\nu-3/2}/\Gamma(\nu-1/2)$ and then integrating the resulting series term by term. The above process which results in (18) is therefore legitimate when $\nu>1/2$, that is, for fractional integration.

When $\nu<1/2$ we have fractional differentiation. Suppose $-1/2<\nu<1/2$; then $p^{-\nu+1/2}$ may be written $p^{-\nu-1/2}p$, which represents first differentiation and then fractional integration. Hence when the Fourier series is of such a character that it can be differentiated term by term, then the process $p^{-\nu-1/2}$, which is fractional integration, can be applied to the resulting Fourier series. That is, the original Fourier series under these conditions can be fractionally differentiated and the result will equal the fractional derivative of the function represented by the series.

A known sufficient condition for term-by-term differentiation of a Fourier series in an interval $-a\leqq x\leqq a$ is that $f(x)$ shall have a derivative and shall be finite and continuous in the interval, and have only a finite number of maxima and minima, and that $f(a)=f(-a)$. Similar criteria will apply for $-3/2<\nu<-1/2$. Here $p^{-\nu+1/2}$ may be written $p^{-\nu-3/2}p^2$ and if the Fourier series may be differentiated twice, the fractional differentiation is legitimate and similar reasoning holds for lower values of ν. Let

$$x^{-\nu/2}p^{-\nu+1/2}f(x^{1/2})=\phi(x^{1/2}).$$

Then (18) becomes when x is substituted for $x^{1/2}$

$$(19)\quad \phi(x)=\frac{f(0)x^{\nu-1}}{\Gamma(\nu+\frac{1}{2})}+\pi^{1/2}\left(\frac{2c}{\pi}\right)^{\nu-1}\sum_{n=1,2\cdots}^{\infty}\frac{1}{n^{\nu-1}}\cdot\left[a_nJ_\nu\left(\frac{n\pi x}{c}\right)-b_nH_\nu\left(\frac{n\pi x}{c}\right)\right],$$

which is a generalized Schlömilch expansion of a function. For the validity of this expansion, the following conditions are sufficient.

(1) The function $\phi(x)$ satisfies the condition that if $f(x^{1/2}) = p^{\nu-1/2}x^{\nu/2}\phi(x^{1/2})$, then $f(x)$ satisfies Dirichlet's conditions sufficient for representation of $f(x)$ by a Fourier series.

(2) $\nu \geqq 1/2$, provided $p^{-\nu+1/2}$ is taken as the identical operation when $\nu = 1/2$.

(3) Or, if $1/2-n<\nu<1/2-(n-1)$, $(n=1, 2, 3, \cdots)$, the function $f(x)$ satisfies the known conditions that $f^n(x)$ will equal the corresponding Fourier series when differentiated term by term n times.

(4) The expansion is true for $-a<x<a$.

If $f(x)$ is represented by a Fourier sine series in the interval $0<x<c$, equation (19) becomes

$$\phi(x) = \pi^{1/2}\left(\frac{2c}{\pi}\right)^{\nu-1} \sum_{n=1,2\cdots}^{\infty} \frac{a_n}{n^{\nu-1}} J_\nu\left(\frac{n\pi x}{c}\right). \tag{20}$$

If $f(x)$ is represented by a Fourier cosine series in the interval $0<x<c$, equation (19) becomes

$$\phi(x) = \frac{f(0)x^{\nu-1}}{\Gamma(\nu+\frac{1}{2})} - \pi^{1/2}\left(\frac{2c}{\pi}\right)^{\nu-1} \sum_{n=1,2\cdots}^{\infty} \frac{b_n}{n^{\nu-1}} H_\nu\left(\frac{n\pi x}{c}\right), \tag{21}$$

where

$$a_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx, \quad b_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx.$$

Hence $\phi(x^{1/2}) = x^{-\nu/2}p^{-\nu+1/2}f(x^{1/2})$ and the same limitations hold as regards $f(x)$ and the values of ν apply as for (19).

EXAMPLE. In (20), let us suppose that $f(x^{1/2}) = 1$ if $0<x<c$, and $f(x^{1/2}) = -1$ if $c<x<2c$. Then

$$\phi(x^{1/2}) = x^{-\nu/2}p^{-\nu+1/2} = \frac{x^{-\nu/2}x^{\nu-1/2}}{\Gamma(\nu+\frac{1}{2})} = \frac{x^{\nu/2-1/2}}{\Gamma(\nu+\frac{1}{2})},$$

and $a_n = 4/(n\pi)$ for $n = 1, 3, 5, \cdots$, and $a_n = 0$ for $n = 2, 4, \cdots$. Hence

$$\phi(x) = \frac{x^{\nu-1}}{\Gamma(\nu+\frac{1}{2})} = \pi^{1/2}\left(\frac{2c}{\pi}\right)^{\nu-1} \sum_{n=1,3,5\cdots}^{\infty} \frac{4}{\pi n^\nu} J_\nu\left(\frac{n\pi x}{c}\right),$$

$$(0 < x < c, \ \nu \geqq \tfrac{1}{2}).$$

Proceeding in exactly the same manner but starting with $f(x^{1/2}) = x^{1/2}/\Gamma(3/2)$, we obtain

$$x^{\nu} = -\sum_{n=1,2\cdots}^{\infty} (-1)^n \left(\frac{2c}{\pi}\right)^{\nu} \frac{2\Gamma(\nu+1)}{n^{\nu}} J_{\nu}\left(\frac{n\pi x}{c}\right), \tag{22}$$

$$(0 < x < c,\ \nu \geqq 0).$$

5. *Examples of Fourier Series with the Derived Schlömilch Series.*

$$\frac{\pi \sin ax}{2 \sin a\pi} = \sum_{n=1,2\cdots}^{\infty} (-1)^{n-1} \frac{n \sin nx}{n^2 - a^2}, \tag{23}$$

$$(-\pi < x < \pi,\ a \text{ not an integer}).$$

$$\frac{\pi a^{1-\nu}}{2 \sin a\pi} J_{\nu}(ax) = \sum_{n=1,2\cdots}^{\infty} (-1)^{n-1} \frac{nJ_{\nu}(nx)}{(n^2 - a^2)n^{\nu-1}},$$

$$(-\pi < x < \pi,\ \nu \geqq \tfrac{1}{2}).$$

$$\frac{\pi \cos ax}{2 \sin a\pi} = \frac{1}{2a} + \sum_{n=1,2\cdots}^{\infty} (-1)^n \frac{a \cos nx}{a^2 - n^2}, \tag{24}$$

$$(-\pi \leqq x \leqq \pi,\ a \text{ not an integer}).$$

$$\frac{\pi H_{\nu}(ax)}{2a^{\nu} \sin a\pi} = \sum_{n=1,2\cdots}^{\infty} (-1)^n \frac{1}{(a^2 - n^2)n^{\nu-1}} H_{\nu}(nx),$$

$$(-\pi \leqq x \leqq \pi,\ \nu \geqq -\tfrac{1}{2}).$$

$$\frac{\pi}{4} \sin x = \tfrac{1}{2} - \sum_{n=1,2\cdots}^{\infty} \frac{\cos 2nx}{(2n-1)(2n+1)}, \quad (0 \leqq x \leqq \pi). \tag{25}$$

$$J_{\nu}(x) = \frac{1}{2^{\nu-3}\pi} \sum_{n=1,2\cdots}^{\infty} \frac{1}{(2n-1)(2n+1)n^{\nu-1}} H_{\nu}(2nx),$$

$$(0 \leqq x \leqq \pi,\ \nu \geqq -\tfrac{1}{2}).$$

$$e^{iax} = \frac{2}{\pi} \sum_{n=1,2\cdots}^{\infty} \frac{n}{n^2 - a^2}(1 - \cos n\pi e^{ia\pi}) \sin nx, \tag{26}$$

$$(0 < x < \pi,\ a \text{ not an integer}),$$

$$\frac{x^{\nu-1}}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})} - \left(\frac{2}{a}\right)^{\nu-1} H_\nu(ax) = -i\left(\frac{2}{a}\right)^{\nu-1} J_\nu(ax)$$

$$+\frac{2}{\pi}\sum_{n=1,2\cdots}^{\infty} \frac{n}{n^2-a^2}(1-\cos n\pi e^{ia\pi})\left(\frac{2}{n}\right)^{\nu-1} J_\nu(nx),$$

where $0<x<\pi$ and $\nu \geqq 1/2$.

(27) $\quad f(x) = a$ if $0 < x < c/2$; $f(x) = 0$ if $x = c/2$;

$$f(x) = -a \text{ if } c/2 < x < c.$$

$$\frac{4a}{\pi}\sum_{n=1,2\cdots}^{\infty} \frac{1}{2n-1}\sin\frac{2(2n-1)\pi}{c}x = f(x),\ (0<x<c),$$

$$\frac{4a}{\pi^{1/2}}\sum_{n=1,2\cdots}^{\infty} \frac{1}{(2n-1)}\left[\frac{c}{(2n-1)\pi}\right]^{\nu-1} J_\nu\left[\frac{2(2n-1)\pi x}{c}\right]$$

$$= \frac{ax^{\nu-1}}{\Gamma(\nu+\frac{1}{2})}, \qquad (0 < x \leqq c/2),$$

$$= \frac{ax^{\nu-1} - 2ax^{-\nu}(x^2-c^2/4)^{\nu-1/2}}{\Gamma(\nu+\frac{1}{2})}, \quad (c/2 \leqq x < c, \nu \geqq \tfrac{1}{2}).$$

It will be noted that the method described in this paper of obtaining the Schlömilch series from a corresponding Fourier series places no restriction on the upper value of ν. In such literature on Schlömilch series as the author has seen, the upper limit of ν is $\nu<1/2$, that is, the order of the Bessel or Struve's function in the series $<1/2$.

St. Louis, Missouri

ON THE INTEGRATION OF UNBOUNDED FUNCTIONS*

BY W. M. WHYBURN

1. *Introduction.* The author has shown† that F. Riesz' treatment of integration‡ leads in every case to the Lebesgue integral. This demonstration makes possible a complete development of the theory of Lebesgue integration from the Riesz point of view.§ Such a development offers a number of advantages over the usual treatment and is especially desirable when one wishes to build a Lebesgue theory on a previous treatment of the Riemann integral. The purpose of the present paper is to emphasize further the importance of the Riesz point of view in a treatment of the general subject of integration by establishing additional relations between sequences of simple functions and functions that are summable in the senses of Lebesgue, Harnack, Denjoy, Denjoy-Khintchine-Young, and Young. The terminology and notation of Riesz' paper‖ are used.

2. *Preliminary Definitions and Theorems.* In this section we give a number of definitions and theorems which are well known but which are essential for the development of later theorems.

Simple function.¶ A function $\phi(x)$ is said to be a simple function on X: $a \leqq x \leqq b$, if there exist $n+1$ points: $x_0 = a < x_1 < x_2 < \cdots < x_n = b$, such that $\phi(x) \equiv \phi_i$, a constant, on $I_i: x_i < x < x_{i+1}$.

Null set. A set of points K is said to be of measure zero if for each $\epsilon > 0$, there exists an at most countably infinite set of intervals such that each point of K is an interior point of some one of these intervals and the sum of the lengths of the intervals

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* Presented to the Society, November 28, 1931.

† This Bulletin, vol. 37(1931), pp. 561–564.

‡ Acta Mathematica, vol. 42(1920), pp. 191–205.

§ This point of view is essentially that used in the ordinary treatment of the Riemann integral.

‖ Loc. cit. We work entirely in the real domain.

¶ These functions have also been called *horizontal* or *step functions*. In this connection see Ettlinger, American Journal of Mathematics, vol. 48 (1926), pp. 215–222.

is less than ϵ. Such a set of points is called a *null set*, and a property which holds everywhere on an interval X with the exception of points of a null set is said to hold *almost everywhere* on X, or is said to hold on X_0.

Integral of a simple function. By $\int_a^b \phi(x)dx$, where $\phi(x)$ is a simple function on X, we mean $\sum_{i=0}^{n-1} \phi_i \Delta_i$, where $\Delta_i = x_{i+1} - x_i$ is the length of I_i.

THEOREM 1. *If $f(x)$ is the limit function on X_0 of a uniformly bounded sequence $[\phi_n(x)]$ of simple functions, then*

$$\lim_{n\to\infty} \int_a^b \phi_n(x)dx$$

exists, and furthermore this limit is the same for all uniformly bounded sequences of simple functions which approach $f(x)$ on X_0.

Theorem 1 was proved by F. Riesz.* He used the limit whose existence is asserted in this theorem as a definition of $\int_a^b f(x)dx$. The author has established† the following theorem which shows that Riesz' integral is identical with the Lebesgue integral.

THEOREM 2. *A necessary and sufficient condition that a bounded function be measurable is that it be the limit function on X_0 of a uniformly bounded sequence of simple functions.*

Riesz established‡ the following theorem.

THEOREM 3. *A necessary and sufficient condition that a bounded function $f(x)$ be Riemann integrable on X is that there exist a sequence $[\phi_n(x)]$ of simple functions which approaches $f(x)$ uniformly almost everywhere on X, that is, uniformly in the neighborhoods of all points on X_0.*

Measure of a set of points. Let K be a set of points on the interval X and let $f(x) = 1$ at points of K and $f(x) = 0$ on $X - K$. If $\int_a^b f(x)dx$ exists§ the point set K is said to be measurable and its measure $m(K)$ is defined as the value of $\int_a^b f(x)dx$.

3. *Unbounded Functions.* We now consider functions which may be unbounded on their interval of definition $X: a \leqq x \leqq b$.

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* Loc. cit., p. 196.

† W. M. Whyburn, loc. cit., pp. 561 and 564.

‡ Loc. cit., p. 204.

§ This is a Lebesgue integral but is defined by the Riesz method and hence uses the measure of null sets only in its definition.

THEOREM 4. *A necessary and sufficient condition that a function $f(x)$ be measurable on X is that there exist a sequence of simple functions that approaches $f(x)$ on X_0.*

PROOF. *Necessity.* Let $f(x)$ be measurable and let $F_i(x)=f(x)$ when $|f(x)|\leqq i$, $F_i(x)=i$ when $f(x)>i$, and $F_i(x)=-i$ when $|f(x)|<-i$. The bounded functions $F_i(x)$, $(i=1, 2, \cdots)$, are measurable and hence by Theorem 2 there exist sequences $[\phi_{in}(x)]$ of simple functions which approach these functions on X_0. By Egeroff's theorem* the sequence $[\phi_{in}(x)]$ approaches $F_i(x)$ uniformly on X except for a set of arbitrarily small measure. For each i choose an index N_i so that $|\phi_{iN_i}(x)-F_i(x)|<(1/2^i)$ except on a set E_i of measure less than $1/2^i$. Let $\phi_i(x)\equiv\phi_{iN_i}(x)$, $(i=1, 2, 3, \cdots)$. We prove that $[\phi_i(x)]$ is a sequence which approaches $f(x)$ on X_0. Let E be the subset of X on which $[\phi_i(x)]$ does not approach $f(x)$. If E is of positive measure, we can determine a number j such that the subset D of E on which $|f(x)|\leqq j$ is of positive measure c (otherwise E would consist of an at most countably infinite set of null sets and would therefore be of measure zero). Let j be chosen and then choose $n>j$ such that $1/2^n<c/4$. We have $|\phi_{n+r}(x)-F_{n+r}(x)|<1/(2^{n+r})$ except on a set of measure less than $1/(2^{n+r})$, $(r=1, 2, \cdots)$. At each point of D, $F_{n+r}(x)=f(x)$, and hence with the possible exception of a subset of measure less than $\sum_{r=1}^{\infty}1/(2^{n+r})<c/2$, we have $|\phi_{n+r}(x)-f(x)|<1/(2^{n+r})$, $(r=1, 2, \cdots)$, on D. Hence $\lim_{j\to\infty}\phi_j(x)=f(x)$ on a subset of D of measure $c/2$ which contradicts the assumption that this limit does not hold at any point of D. This completes the proof of the necessity.

Sufficiency. Let $[\phi_n(x)]$ be a sequence of simple functions which approaches $f(x)$ on X_0 and let M_1 and M_2, $M_1<M_2$, be any two constants. Let E be the subset of X on which $M_1\leqq f(x)\leqq M_2$. Let $F(x)=f(x)$ when $M_1-1\leqq f(x)\leqq M_2+1$, $F(x)=M_1-1$ when $f(x)<M_1-1$, $F(x)=M_2+1$ when $f(x)>M_2+1$. Let $\theta_i(x)=\phi_i(x)$ when $\phi_i(x)$ lies between M_1-1 and M_2+1, $\theta_i(x)=M_1-1$ when $\phi_i(x)\leqq M_1-1$, $\theta_i(x)=M_2+1$ when $\phi_i(x)\geqq M_2+1$. Now $[\theta_i(x)]$ is a uniformly bounded sequence of simple functions which approaches the bounded function $F(x)$ on X_0. Hence by Theorem 2, $F(x)$ is measurable on X and the set of points for which $M_1\leqq F(x)\leqq M_2$ is measurable. This set, how-

* See Hobson, *Functions of a Real Variable*, vol. 2, p. 140.

ever, is the set E and hence E is measurable. It follows from this that $f(x)$ is measurable on X.

Definition of property B. A sequence $[\phi_n(x)]$ of simple functions is said to have property B if for each $\epsilon>0$, there exists a number M_ϵ such that if $I(n, \epsilon)$ denotes the set of intervals on which $|\phi_n(x)|>M_\epsilon$, then $\int_{I(n,\epsilon)}|\phi_n(x)|dx<\epsilon$, $(n=1, 2, \cdots)$, where $\int_{I(n,\epsilon)}|\phi_n(x)|dx$ means the sum of the integrals of $|\phi_n(x)|$ on the separate intervals of $I(n, \epsilon)$.

THEOREM 5. *If $[\phi_n(x)]$ is a sequence of simple functions which has property B and which is such that $\lim_{n\to\infty}\phi_n(x)$ exists on X_0, then $\lim_{n\to\infty}\int_a^b\phi_n(x)dx$ exists.*

PROOF. Let $\epsilon=1$. We know that $\left|\int_a^b\phi_n(x)dx\right|\leqq M_1(b-a)+\int_{I(n,1)}|\phi_n(x)|dx<M_1(b-a)+1$ for all n. The set of numbers $K_n=\int_a^b\phi_n(x)dx$ is therefore bounded. Hence there is at least one number K which either is identical with infinitely many of the numbers K_n or is the limit of a subsequence of $[K_n]$. In either case we may pick out a sequence $[\theta_i(x)]$ of simple functions from the sequence $[\phi_n(x)]$ in such a way that $[\theta_i(x)]$ approaches the same limit function $f(x)$ as $[\phi_n(x)]$, has property B, and is such that $\lim_{i\to\infty}\theta_i(x)dx$ exists.

LEMMA. *If $[h_n(x)]$ and $[g_n(x)]$ are two sequences of simple functions which approach $f(x)$ on X_0 and which have property B, then if $\lim_{n\to\infty}\int_a^b h_n(x)dx$ and $\lim_{n\to\infty}\int_a^b g_n(x)dx$ exist, these limits are equal.*

PROOF OF LEMMA. Let us suppose that $\lim_{n\to\infty}\int_a^b h_n(x)dx=H$, $\lim_{n\to\infty}\int_a^b g_n(x)dx=G$, and let $\eta=7\epsilon>0$ be arbitrarily assigned. Choose N_1 so that for all $n>N_1$

$$(1)\qquad \left|\int_a^b h_n(x)dx-H\right|<\epsilon,\qquad \left|\int_a^b g_n(x)dx-G\right|<\epsilon.$$

Let M be the larger of the two numbers M_ϵ for $[h_n(x)]$ and $[g_n(x)]$ by property B. Let $J(n, \epsilon)$ and $J'(n, \epsilon)$, respectively, be the sets of intervals on which $|g_n(x)|>M$ and $|h_n(x)|>M$. Hence

$$\int_{J(n,\epsilon)}|g_n(x)|dx<\epsilon,\quad \int_{J'(n,\epsilon)}|h_n(x)|dx<\epsilon,\ (n=1, 2, \cdots).$$

Let $G_n(x)=g_n(x)$ when $|g_n(x)|\leqq M$, $G_n(x)=M$ when $g_n(x)>M$,

$G_n(x) = -M$ when $g_n(x) < -M$; and let $H_n(x) = h_n(x)$ when $|h_n(x)| \leqq M$, $H_n(x) = M$ when $h_n(x) > M$, $H_n(x) = -M$ when $h_n(x) < -M$. It follows immediately that $[|G_n(x) - H_n(x)|]$ is a sequence of uniformly bounded simple functions that approaches zero on X_0. By Theorems 1 and 2 we may choose an index N, $N > N_1$, such that for all $n > N$

$$\int_a^b |G_n(x) - H_n(x)|\, dx < \epsilon. \tag{4}$$

We have

$$\int_a^b G_n(x)dx - \int_{J(n,\epsilon)} [M + |g_n(x)|]dx \leqq \int_a^b g_n(x)dx$$

$$\leqq \int_a^b G_n(x)dx + \int_{J(n,\epsilon)} [M + |g_n(x)|]dx,$$

$$\int_a^b H_n(x)dx - \int_{J'(n,\epsilon)} [M + |h_n(x)|]dx \leqq \int_a^b h_n(x)dx$$

$$\leqq \int_a^b H_n(x)dx + \int_{J'(n,\epsilon)} [M + |h_n(x)|]dx.$$

Since $M < |g_n(x)|$ on $J(n, \epsilon)$ and $M < |h_n(x)|$ on $J'(n, \epsilon)$, we have

$$\int_a^b G_n(x)dx - 2\epsilon < \int_a^b g_n(x)dx < \int_a^b G_n(x)dx + 2\epsilon, \tag{2}$$

$$\int_a^b H_n(x)dx - 2\epsilon < \int_a^b h_n(x)dx < \int_a^b H_n(x)dx + 2\epsilon. \tag{3}$$

If we combine (2) and (3) and make use of (4), we obtain

$$\left| \int_a^b g_n(x)dx - \int_a^b h_n(x)dx \right| < \left| \int_a^b [G_n(x) - H_n(x)]dx \right| + 4\epsilon < 5\epsilon, \text{ for } n > N. \tag{5}$$

A combination of (5) with (1) yields $|G - H| < 7\epsilon$, for all $n > N$. Since $\eta = 7\epsilon$ was arbitrarily chosen, it follows that $G = H$.

PROOF OF THEOREM 5. If $\lim_{n\to\infty} \int_a^b \phi_n(x)dx$ is not K, we may pick a subsequence $[g_j(x)]$ from $[\phi_n(x)]$ which approaches $f(x)$

on X_0, has property B, and is such that $\lim_{j\to\infty} \int_a^b g_j(x)dx$ exists and is different from K. But by the lemma, $\int_a^b g_j(x)dx$ must approach the same limit as $\int_a^b \theta_i(x)dx$. It follows from this that $\lim_{n\to\infty} \int_a^b \phi_n(x)dx$ exists.

COROLLARY. *If* $[g_n(x)]$ *and* $[h_n(x)]$ *are any two sequences of simple functions that approach* $f(x)$ *on* X_0 *and that have property* B, *then* $\lim_{n\to\infty} \int_a^b g_n(x)dx$ *and* $\lim_{n\to\infty} \int_a^b h_n(x)dx$ *exist and these limits are equal.*

THEOREM 6. *A necessary and sufficient condition that a function* $f(x)$ *be summable (in the sense of Lebesgue) is that there exist a sequence* $[\phi_n(x)]$ *of simple functions which approaches* $f(x)$ *on* X_0 *and which has property* B.

Proof of sufficiency. This may be proved directly or we may obtain it as a consequence of a theorem due to de la Vallée Poussin.* We note that the absolute continuity of the integrals $\int \phi_n(x)dx$, $(n=1, 2, \cdots)$, is uniform,† since for a given $\epsilon>0$ we may choose $\delta=\epsilon/[2M_{\epsilon/2}]$, and δ is independent of n. It follows then from de la Vallée Poussin's theorem that $f(x)$ is summable on X.

Proof of necessity. Let $f(x)$ be summable on X and let X be subdivided into n equal parts by the points $x_1, x_2, \cdots, x_{n-1}$. Let $x_0=a$ and $x_n=b$. On $I_i: x_i \leqq x < x_{i+1}$, let $\phi_n(x)=\int_{x_i}^{x_{i+1}} f(x)dx/\Delta_i$, $(i=0, \cdots, n-1)$, where $\Delta_i=x_{i+1}-x_i$. It follows‡ from the absolute continuity of $\int f(x)dx$ that $\lim_{n\to\infty} \phi_n(x)=f(x)$ on X_0. It remains to show that $[\phi_n(x)]$ has property B. Let $g(x)=|f(x)|$, $h_n(x)=\int_{x_i}^{x_{i+1}} g(x)dx/\Delta_i$ on I_i, $(i=0, 1, \cdots, n-1)$. Let $\epsilon>0$ be assigned and let $\delta>0$ be chosen so that $\int_{(e)} g(t)dt<\epsilon$, when e is a measurable subset of X such that $m(e)\leqq\delta$. The existence of δ follows from the absolute continuity of $\int_a^x g(t)dt$. Let $g_j(x)=g(x)$ when $|g(x)|\leqq j$, $g_j(x)=j$ when $g(x)>j$. The integral $\int_a^b g_j(x)dx$ is an increasing function of j and has $\int_a^b g(x)dx$ as its limit as j becomes infinite. Choose J so that for all $j\geqq J$, $\int_a^b g(x)dx - \int_a^b g_j(x)dx<\epsilon$. Let k denote the subset of X on which $g(x)>J$ and let k_i denote the subset of k that lies on I_i. We have $\int_{(k)} g(x)dx=\sum_{i=0}^{n-1}\int_{(k_i)} g(x)dx<\epsilon$. Now

* Transactions of this Society, vol. 16 (1915), p. 445.

† For definition, see de la Vallée Poussin, loc. cit. p. 445.

‡ See W. M. Whyburn, loc. cit., p. 562, for a detailed proof of this.

$$h_n(x) = [\int_{(I_i - k_i)} g(x)dx + \int_{(k_i)} g(x)dx]/\Delta_i$$
$$\leqq J + \int_{(k_i)} g(x)dx/\Delta_i, \text{ on } I_i.$$

Let $M_\epsilon = J + \epsilon/\delta$. In order that $h_n(x)$ be greater than M_ϵ on I_i it is necessary that $\int_{(k_i)} g(x)dx/\Delta_i$ be greater than ϵ/δ and hence Δ_i be less than $[\delta/\epsilon]\int_{(k_i)} g(x)dx$. From this it follows that the sum $I'(n, \epsilon)$ of intervals on which $h_n(x) > M_\epsilon$ is of measure less than $\sum_{i=0}^{n-1} [\delta/\epsilon]\int_{(k_i)} g(x)dx < \epsilon\delta/\epsilon \leqq \delta$. Hence $\int_{I'(n,\epsilon)} g(x)dx = \int_{I'(n,\epsilon)} h_n(x)dx < \epsilon$. Since $|\phi_n(x)| \leqq h_n(x)$, it follows that the set $I(n, \epsilon)$ of intervals on which $|\phi_n(x)| > M_\epsilon$ is a subset of $I'(n, \epsilon)$ and that $\int_{I(n,\epsilon)} |\phi_n(x)|dx$ is less than ϵ for all n. This proves that $[\phi_n(x)]$ has property B.

THEOREM 7. *If $f(x)$ is summable on X and if $[\phi_n(x)]$ is any sequence of simple functions which has property B and which approaches $f(x)$ on X_0, then $\lim_{n\to\infty} \int_a^b \phi_n(x)dx$ exists and is equal to $\int_a^b f(x)dx$.*

PROOF. This theorem follows immediately from Theorem 5 and a theorem of de la Vallée Poussin.*

A theorem of de la Vallée Poussin† enables us to state Theorem 6 in the following form.

THEOREM 8. *A necessary and sufficient condition that a function $f(x)$ be summable on X: $a \leqq x \leqq b$, is that there exist a sequence $[\phi_n(x)]$ of simple functions that approaches $f(x)$ on X_0 and is such that the absolute continuity of the indefinite integrals of these simple functions is uniform.*

We may now state and prove the following theorems which apply when the integrals are taken in senses that are more general than that of Lebesgue. Let $f(x)$ be a function on $X: a \leqq x \leqq b$, and let any definition of the integral of $f(x)$ be given which exists for $f(x)$ and which has the further property that if $J(a, b)$ denotes this integral and c, d, e are any three points of X, then $J(c, d)$, $J(c, e)$, $J(e, d)$ exist and $J(c, d) = J(c, e) + J(e, d)$.

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* Loc. cit., p. 446, Theorem 2. We note that the proof of Theorem V might be shortened by the use of de la Vallée Poussin's theorems. It seems desirable, however, to give this proof from the Riesz point of view in order that the theorem be available for use in a treatment of Lebesgue integration by the Riesz approach.

† Loc. cit., p. 450.

THEOREM 9. *Let E be an everywhere dense set of points on X and let M denote the subset of X which has the property that if p is any point of M, then $\lim_{q\to p} J(p, q)/(q-p)$ exists and equals $F(p)$, where q ranges on the set E. There exists a sequence $[\phi_n(x)]$ of simple functions such that this sequence converges to $F(x)$ at each point of M and $\int_a^b \phi_n(x)dx = J(a, b)$ for $n=1, 2, \cdots$.*

PROOF. Let a method of subdivision of X be chosen in such a way that all of the subdivision points belong to E and the length of the maximum subdivision approaches zero as the number of these subdivisions becomes infinite. Let the points of subdivision, in order of increasing magnitude, for the nth stage of the subdivision be $x_0=a, x_1, x_2, \cdots, x_n=b$. On I_i: $x_i \leqq x < x_{i+1}$, let $\phi_n(x) = J(x_i, x_{i+1})/(x_{i+1}-x_i)$, $(i=0, 1, \cdots, n-1)$. Consider $[\phi_n(x)], (n=1, 2, \cdots)$. Let p be any point of M, and for each n let I_{p_n}: $p_n \leqq x < q_n$ be the subdivision that contains p. Now

$$\begin{aligned}\phi_n(p) &= J(p_n, q_n)/(q_n-p_n)\\ &= [J(p_n, p)+J(p, q_n)]/(q_n-p_n) = J(p_n, p)/(p-p_n)\\ &+ [J(p, q_n)/(q_n-p) - J(p_n, p)/(p-p_n)]\,[(q_n-p)/(q_n-p_n)].\end{aligned}$$

We note that $|(q_n-p)/(q_n-p_n)| \leqq 1$ while its coefficient has zero for its limit (since both terms approach $F(p)$). Since $\lim_{n\to\infty} J(p_n, p)/(p-p_n)$ is $F(p)$, we have $\lim_{n\to\infty} \phi_n(p) = F(p)$. In case p is a point of subdivision,* we replace the indeterminate ratio in the above formula for $\phi_n(p)$ by the limit of this ratio (which is $F(p)$). Finally, we note that, for each n,

$$\int_a^b \phi_n(x)dx = \sum_{i=0}^{n-1} J(x_i, x_{i+1}) = J(a, b)$$

COROLLARY 1. *If E is identical with X, then M is the set of points at which $J(a, x)$ has a derivative with respect to x. In particular, if this derivative is equal to $f(x)$ at a point of M, then $[\phi_n(x)]$ converges to $f(x)$ at this point.*

COROLLARY 2. *If $f(x)$ is integrable in the sense of Harnack-Lebesgue, Denjoy, or Lebesgue, the conditions of Theorem 9 are satisfied if E is any everywhere dense set on X and M is X_0. Furthermore, $F(x) \equiv f(x)$ on X_0.*

COROLLARY 3. *If $f(x)$ is integrable in the Denjoy-Khintchine-Young sense and if X can be divided into a countable number of*

* We define $\phi_n(b) = \phi_n(x_{n-1})$.

measurable sets in such a way that the indefinite integrals (in the D-K-Y sense) are absolutely continuous on each of these, then the conditions of Theorem 9 are satisfied when $J(a, b)$ denotes the D-K-Y integral of $f(x)$, E is any everywhere dense set on X, M is X_0, and $F(x) \equiv f(x)$ on X_0.*

It is a very simple matter to extend the results of the present paper to cases where the interval of definition X is replaced by any measurable point set E on X. The definition of $f(x)$ is extended to points of $X-E$ by letting $f(x)$ be zero at such points. The integral of $f(x)$ on E is then described in terms of the integral of the extended function on X. One could define a simple function on a point set by saying that it is a function which takes on only a finite number of values on this set. This definition is not needed in the present adaptation.

THE UNIVERSITY OF CALIFORNIA AT LOS ANGELES

REFLECTIONS IN FUNCTION SPACE†

BY L. S. KENNISON

1. *Introduction.* The purpose of this paper is to point out an error,‡ giving the corrected form of the incorrect theorem referred to below as Delsarte's theorem, also to prove a generalization. However, the contribution made to the geometry of function space may be interesting to some on its own account.

We shall consider only functions of one or two variables defined on the interval (a, b) or the corresponding square. All such functions are to be bounded and integrable on the range of definition. We shall denote the continuous arguments on (a, b) by the letters x, s, t, u, written as subscripts or superscripts and shall imply integration on (a, b) with respect to any argument that occurs twice in the same term.

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* See Khintchine, Comptes Rendus, vol. 152 (1916), p. 290.

† Presented to the Society, February 28, 1931.

‡ See §3 below.

2. *Functional Rotations and Reflections.* Delsarte* has defined functional rotations as follows. The functional transformation†

$$y^z = \bar{y}^z + K_s^z \bar{y}^s, \tag{1}$$

from $\bar{y}^z$ to y^z, is a functional rotation if and only if

$$K_s^z + K_z^s + K_z^u K_s^u = 0, \tag{2}$$

$$K_s^z + K_z^s + K_u^z K_u^s = 0. \tag{3}$$

Delsarte shows that these two conditions are equivalent. A kernel such that these are satisfied is called a *kernel of rotation.* From Fredholm's product theorem we see that

$$D_k(-1) = \pm 1, \tag{4}$$

where $D_k(\lambda)$ is defined in the usual manner. Delsarte‡ states and attempts to prove the following theorem, which we shall call Delsarte's Theorem.

The necessary and sufficient condition that a kernel be one of rotation is that the kernel be the value taken by the resolvent of a skew-symmetric kernel for the value $\lambda = 1/2$ *of the parameter.*

Let $f^z = f_z$ be a function such that $f^z f_z = 1$, and consider the functional transformation

$$\bar{y}^z = y^z - 2f^z f_s y^s. \tag{5}$$

Denote by g_1 the Fourier coefficient of an arbitrary function g^z with respect to the set of one normal and orthogonal function f^z. Then we may write

$$g^z = g_1 f^z + r^z,$$

where $r^s f_s = 0$. Applying (5), we have

$$\bar{g}^z = -g_1 f^z + r^z.$$

Hence the transformation (5) reverses the component of g^z in the direction of f^z and leaves unchanged the component r^z orthogonal to f^z. In other words, the function $\bar{g}^z$ may be thought of

* Annales de Toulouse, vol. 20 (1928), p. 47 ff.

† The corresponding integral equation, in Delsarte's notation, is of the form $\bar{y}(x) = y(x) + \lambda \int_a^b K(x, s)\bar{y}(s)ds$, which reduces to (1) for $\lambda = -1$.

‡ Loc. cit., p. 56.

as the *reflection* of g^z in the $(\infty - 1)$-dimensional variety orthogonal to f^z and containing the origin of function space. Accordingly we call (5) a *functional reflection.*

3. *Proper and Improper Rotations.* Now call any kernel of rotation that is the resolvent of a skew-symmetric kernel with parameter $\frac{1}{2}$ a *proper* kernel of rotation, and the corresponding transformation (1) a *proper functional rotation.* Any other kernel of rotation will be called an *improper* kernel of rotation, and its transformation an *improper functional rotation.* Delsarte's theorem states that every kernel of rotation is proper.

THEOREM 1. *Every functional reflection is an improper functional rotation.*

It is easy to verify that (2) and (3) are satisfied for the kernel $-2f^z f_s$. It is also seen by a little calculation that the resolvent of a skew-symmetric kernel cannot be symmetric for any value of the parameter. Hence $-2f^z f_s$ is an improper kernel of rotation. This contradiction with Delsarte's theorem is explained by the following mistake in Delsarte's proof. On page 55 (loc. cit.) Delsarte defines the kernel $h(s, t)$ as $\Gamma(s, t; -1/2)$, where $\Gamma(s, t; \lambda)$ is the resolvent kernel of $K(s, t)$, the kernel of rotation in question. He then proves that $h(s, t)$ is skew-symmetric and that $K(s, t)$ is the value of the resolvent of $h(s, t)$ for $\lambda = 1/2$. If $-1/2$ is a pole of $\Gamma(s, t; \lambda)$, then $h(s, t)$ is not defined, and Delsarte's theorem fails to hold. It is readily seen that the resolvent of the kernel $-2f^z f_s$ is

$$-\frac{2f^z f_s}{1 + 2\lambda},$$

for which $\lambda = -1/2$ is a pole. With this mistake corrected, Delsarte's theorem takes the following form:

THEOREM 2. *A kernel of rotation, for which* $-1/2$ *is not a pole of the resolvent, is a proper kernel of rotation.*

The generalization of Theorem 2 is as follows.

THEOREM 3. *Let* K_z^x *be a kernel of rotation, such that* $-1/2$ *is an n-fold root of its Fredholm determinant* $D(\lambda)$. *Then the functional transformation* (1) *corresponding to* K_z^x *is the product of n functional reflections followed by a proper functional rotation.*

LEMMA 1. *Let K_s^x be a kernel of rotation, such that $-1/2$ is an r-fold root of its Fredholm determinant. Then the functional transformation* (1) *corresponding to K_s^x is the product of a functional reflection and a rotation for whose kernel $-1/2$ is an $(r-1)$-fold root of the Fredholm determinant.*

Let ϕ^x be a normalized fundamental function of K_s^x, corresponding to $\lambda=-1/2$. Consider the kernel $L_s^x=-2\phi^x\phi^s$ and $R_s^x=K_s^x-L_s^x$. Then $K_s^x\phi^s=-2\phi^x$ and we have

$$L_s^x + R_s^x + R_t^x L_s^t = K_s^x,$$

and so $D_K(\lambda)=(1+2\lambda)\,D_R(\lambda)$.

Direct substitution of $K_s^x+2\phi^x\phi_s$ in (3) shows that (3) is satisfied for R_s^x. Multiplying (2) by ϕ^s, integrating, and using $K_s^x\phi^s=-2\phi^x$, we obtain $K_s^x\phi_x=-2\phi_s$, which is sufficient to show that (2) is satisfied for R_s^x. Hence R_s^x is a kernel of rotation and the lemma is proved. Applying the lemma n times, we obtain the theorem.

The formula

$$D_K(\lambda) = (1+2\lambda)^n D_P(\lambda) \tag{6}$$

is also obtained by a repetition of the lemma, where $D_P(\lambda)$ is the determinant of the proper kernel of rotation, P_s^x, of Theorem 3.

LEMMA 2. *The Fredholm determinant $D(-1)$ of a proper functional rotation is $+1$.*

Let P_s^x be a proper kernel of rotation. Then, by Theorem 2, there exists a skew-symmetric kernel H_s^x of which P_s^x is the resolvent for $\lambda=1/2$. The Fredholm determinant of H_s^x is positive for all real λ, since it is $+1$ for $\lambda=0$, is continuous in λ, and has no real roots. Also $D_H(1/2)\,D_P(-1)=1$; hence $D_P(-1)$ is positive, and by (4) is $+1$. Setting $\lambda=-1$ in (6), we find the following theorem.

THEOREM 4. *The determinant $D_K(-1)$ of the transformation* (1), *is equal to $(-1)^n$, where n is the multiplicity of the root $-1/2$ of $D_K(\lambda)$.*

PRINCETON UNIVERSITY

AN EXTENSION OF LAGRANGE'S EQUATIONS

BY C. A. SHOOK

1. *Introduction.* In studying the motion of a planet about the sun under the action of a disturbing planet the following coordinates are convenient. Let the plane determined by the origin and the tangent to the planet's path at any instant be π. Let x, P, N denote respectively the points where a unit sphere about the origin is cut by the x axis, the radius vector to the planet, and the line of intersection of π and the xy plane. Then let $\theta = xN$, $w = NP$, $\phi =$ the dihedral angle between π and the xy plane, and $r =$ radius vector, it being understood that ϕ is in the first quadrant and θ is between 0 and 360°.

If β and λ denote latitude and longitude respectively, we have for the kinetic energy of a unit particle

$$2T = \dot{r}^2 + r^2(\dot{\beta}^2 + \cos^2\beta\dot{\lambda}^2). \tag{1}$$

But we have the relations

$$\left.\begin{aligned} \sin\beta &= \sin\phi \sin w \\ \tan(\lambda - \theta) &= \cos\phi \tan w \end{aligned}\right\}, \tag{2}$$

whence

$$2T = \dot{r}^2 + r^2[(1 - \sin^2 w \sin^2\phi)\dot{\theta}^2 + \dot{w}^2 + \sin^2 w\dot{\phi}^2 + 2\cos\phi\dot{\theta}\dot{w} - 2\sin w \cos w \sin\phi\dot{\theta}\dot{\phi}]. \tag{3}$$

It will be shown that the condition that the velocity be in the plane π is

$$\frac{\partial T}{\partial \dot{\phi}} = 0,$$

which in the present case reduces to

$$\sin w\dot{\phi} - \cos w \sin\phi\dot{\theta} = 0. \tag{4}$$

If by means of this relation we eliminate $\dot{\phi}$ from T we get

$$2T = \dot{r}^2 + r^2(\dot{w} + \cos\phi\dot{\theta})^2.$$

Now put $v = w + \theta$ and $\Gamma = 1 - \cos\phi$, so that

$$2T = \dot{r}^2 + r^2(\dot{v} - \Gamma\dot{\theta})^2. \tag{5}$$

The four coordinates used are r, θ, v, Γ.

The question now is whether or not we may use Lagrange's equations with this value of T. There are two reasons why it is by no means obvious that we may. In the first place we are using four coordinates and the dynamical system has but three degrees of freedom. In the second place, partial derivatives are involved in Lagrange's equations. Now the partial derivatives of T depend not only on its value but on the way it is expressed. We might, for example, have eliminated $\dot{\theta}$ by means of (4); or we might have assumed a relation altogether different from (4). We shall prove a theorem which shows that, in spite of all this, Lagrange's equations do apply in the present case.

2. *Conservative Holonomic Dynamical System.* In a conservative holonomic dynamical system with n degrees of freedom let the kinetic energy, T, be given by

$$2T = a_{ij}\dot{q}_i\dot{q}_j, \tag{6}$$

the double appearance of i and j denoting summation from 1 to n. The a_{ij} are functions of $q_1, \cdots, q_n$ and we may suppose that $a_{ij}=a_{ji}$. Let the kinetic potential be

$$L = T + F(q_1, \cdots, q_n), \tag{7}$$

and let

$$D_{q_i} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\right) - \frac{\partial}{\partial q_i}.$$

With this notation Lagrange's equations are

$$D_{q_i}L = 0.$$

Suppose that the q_i are related to $n+1$ new variables, $r_1, \cdots, r_{n+1}$, by the equations

$$q_i = q_i(r_1, \cdots, r_{n+1}), \qquad (i = 1, 2, \cdots, n), \tag{8}$$

which are such that

$$\frac{\partial(q_1, \cdots, q_n)}{\partial(r_1, \cdots, r_n)} \not\equiv 0.$$

We have

$$\dot{q}_i = \frac{\partial q_i}{\partial r_s}\dot{r}_s, \quad (s = 1, 2, \cdots, n+1), \tag{9}$$

and

$$\frac{\partial \dot{q}_i}{\partial \dot{r}_s} = \frac{\partial q_i}{\partial r_s}. \tag{10}$$

When L is expressed in terms of $r_s, \dot{r}_s$, we shall write it L'. Now

$$\frac{\partial L'}{\partial \dot{r}_s} = \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \dot{r}_s} = \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial r_s},$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{r}_s}\right) = \frac{\partial q_i}{\partial r_s}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\left(\frac{\partial q_i}{\partial r_s}\right),$$

$$\frac{\partial L'}{\partial r_s} = \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial r_s} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial r_s}.$$

Since

$$\frac{\partial \dot{q}_i}{\partial r_s} = \frac{d}{dt}\left(\frac{\partial q_i}{\partial r_s}\right),$$

we have

$$D_{r_s}L' = \frac{\partial q_i}{\partial r_s} D_{q_i}L. \tag{11}$$

Since $D_{q_i}L = 0$ along the trajectories, we have

$$D_{r_s}L' = 0 \tag{12}$$

along these trajectories. But as we should expect, these equations are not sufficient to determine $r_1, \cdots, r_{n+1}$. In fact the equations are not independent, for it can be verified that

$$\sum_{s=1}^{n+1}(-1)^s \frac{\partial(q_1, \cdots, q_n)}{\partial(r_1, \cdots, r_{s-1}, r_{s+1}, \cdots, r_{n+1})} D_{r_s}L' = 0. \tag{13}$$

Let us restrict the variables $r_1, \cdots, r_{n+1}$ by an equation of the form

$$\alpha_s \dot{r}_s = 0, \quad (s = 1, 2, \cdots, n+1, \ \alpha_{n+1} \not\equiv 0), \tag{14}$$

where the α's are functions of $r_1, \cdots, r_{n+1}$. This enables us to eliminate $\dot{r}_{n+1}$ from L'. After this elimination the kinetic po-

tential will be denoted by L''. We note that $L=L'=L''$, but that the partial derivatives of the last two are not in general equal.

If the subscript zero denotes that after differentiation $\dot{r}_{n+1}$ is eliminated by means of (14), then if $s\neq n+1$,

$$\frac{\partial L''}{\partial \dot{r}_s}=\left(\frac{\partial L'}{\partial \dot{r}_s}\right)_0+\left(\frac{\partial L'}{\partial \dot{r}_{n+1}}\right)_0\frac{\partial \dot{r}_{n+1}}{\partial \dot{r}_s},$$

$$\frac{\partial L''}{\partial r_s}=\left(\frac{\partial L'}{\partial r_s}\right)_0+\left(\frac{\partial L'}{\partial \dot{r}_{n+1}}\right)_0\frac{\partial \dot{r}_{n+1}}{\partial r_s}.$$

Hence,

$$D_rL''=(D_rL')_0+\frac{d}{dt}\left[\left(\frac{\partial L'}{\partial \dot{r}_{n+1}}\right)_0\frac{\partial \dot{r}_{n+1}}{\partial \dot{r}_s}\right]-\left(\frac{\partial L'}{\partial \dot{r}_{n+1}}\right)_0\frac{\partial \dot{r}_{n+1}}{\partial r_s}. \tag{15}$$

Let us suppose that (14) is so chosen that

$$\left(\frac{\partial L'}{\partial \dot{r}_{n+1}}\right)_0\equiv 0. \tag{16}$$

It will be seen by Theorem 2A that this assumption is equivalent to saying that (14) is identical with

$$\frac{\partial L'}{\partial \dot{r}_{n+1}}=0. \tag{14'}$$

From (15) and (16) it follows that

$$D_rL''=0 \tag{17}$$

along the trajectories. It is easily shown that this equation also holds for $s=n+1$, in which case it becomes $\partial L''/\partial r_{n+1}=0$.

If we replace L' by L'' in (13), this equation holds by virtue of (15) and (16). It is no longer an identity, however, since it holds only by virtue of (16), that is, of (14). In other words (14) is deducible from equations (17) and may be used in place of any one of these. We can now state the following theorem.

THEOREM 1. *If $L(q_1,\cdots,q_n,\dot{q}_1,\cdots,\dot{q}_n)$ is the kinetic potential in a conservative holonomic dynamical system and we put*

$$q_i=q_i(r_1,\cdots,r_{n+1}) \tag{8}$$

and suppose that

$$\frac{\partial L'}{\partial \dot{r}_{n+1}} = 0, \tag{14'}$$

then $r_1, \cdots, r_{n+1}$ *are determined by the equations*

$$D_s L'' = 0, \tag{17}$$

and (14′) *is a consequence of these equations.*

3. *Alternative Conditions to Replace* (16).

THEOREM 2. *Condition* (16) *implies and is implied by either of the following:*

(A). *The ratio*

$$\alpha_s : a_{ij} \frac{\partial q_i}{\partial r_s} \frac{\partial q_j}{\partial r_{n+1}}$$

is the same for every $s = 1, 2, \cdots, n+1$.

(B). *If* $\dot{r}_{n+1}$ *is eliminated from* $\dot{q}_i$ *by* (14),

$$a_{ij} \frac{\partial q_j}{\partial r_{n+1}} \dot{q}_i \equiv 0.$$

The proof of these conditions is immediate. Since L contains $\dot{q}_i$ and hence $\dot{r}_s$ only through T, (16) becomes

$$\left(\frac{\partial T'}{\partial \dot{r}_{n+1}}\right)_0 \equiv 0. \tag{16'}$$

By (6) and (9) we have

$$2T' = a_{ij} \frac{\partial q_i}{\partial r_s} \frac{\partial q_j}{\partial r_\sigma} \dot{r}_s \dot{r}_\sigma, \qquad (s, \sigma = 1, 2, \cdots, n+1),$$

so that

$$\frac{\partial T'}{\partial \dot{r}_{n+1}} = a_{ij} \frac{\partial q_i}{\partial r_s} \frac{\partial q_j}{\partial r_{n+1}} \dot{r}_s, \qquad (s = 1, 2, \cdots, n+1),$$

and

$$\left(\frac{\partial T'}{\partial \dot{r}_{n+1}}\right)_0 = a_{ij} \frac{\partial q_j}{\partial r_{n+1}} \left[\frac{\partial q_i}{\partial r_k} - \frac{\alpha_k}{\alpha_{n+1}} \frac{\partial q_i}{\partial r_{n+1}}\right] \dot{r}_k \equiv 0, \tag{18}$$

$$(k = 1, \cdots, n).$$

From this it follows that

$$a_{ij}\frac{\partial q_j}{\partial r_{n+1}}\left[\frac{\partial q_i}{\partial r_k}-\frac{\alpha_k}{\alpha_{n+1}}\frac{\partial q_i}{\partial r_{n+1}}\right]\equiv 0, \qquad (k=1,\cdots,n), \tag{19}$$

and hence

$$\alpha_k : a_{ij}\frac{\partial q_i}{\partial r_k}\frac{\partial q_j}{\partial r_{n+1}}=\alpha_{n+1} : a_{ij}\frac{\partial q_i}{\partial r_{n+1}}\frac{\partial q_j}{\partial r_{n+1}}, \tag{20}$$

showing that the ratio on the left is the same for all values of k. Conversely if (20) is satisfied then (19) follows and hence (18), or (16). This proves the equivalence of (A) and (16).

When $\dot{r}_{n+1}$ is eliminated from $\dot{q}_i$ by (14) we get

$$\dot{q}_i=\left[\frac{\partial q_i}{\partial r_k}-\frac{\alpha_k}{\alpha_{n+1}}\frac{\partial q_i}{\partial r_{n+1}}\right]\dot{r}_k, \qquad (k=1,\cdots,n), \tag{21}$$

and hence by (18)

$$a_{ij}\frac{\partial q_j}{\partial r_{n+1}}\dot{q}_i\equiv 0.$$

This proves the equivalence of (B) and (16).

THEOREM 3. *If* (16) *is satisfied, then when* $\dot{r}_{n+1}$ *is eliminated from* $\dot{q}_1,\cdots,\dot{q}_n$,

$$\frac{\partial(\dot{q}_1,\cdots,\dot{q}_n)}{\partial(\dot{r}_1,\cdots,\dot{r}_n)}\equiv 0.$$

Equations (19) may be regarded as n equations for the determination of $\partial q_j/\partial r_{n+1}$. Since these derivatives do not all vanish and the equations defining them are homogeneous, it follows that the determinant

$$|b_{jk}|\equiv\left|a_{ij}\left(\frac{\partial q_i}{\partial r_k}-\frac{\alpha_k}{\alpha_{n+1}}\frac{\partial q_i}{\partial r_{n+1}}\right)\right|$$

is zero. But by the rule for multiplying determinants

$$|b_{jk}|=|a_{ij}|\left|\frac{\partial q_i}{\partial r_k}-\frac{\alpha_k}{\alpha_{n+1}}\frac{\partial q_i}{\partial r_{n+1}}\right|, \quad (i \text{ not summed}),$$

$$=|a_{ij}|\left|\frac{\partial\dot{q}_i}{\partial\dot{r}_k}\right|, \quad (i \text{ not summed}),$$

by (21). Since $|a_{ij}|$ is the determinant of a positive quadratic form, we must have

$$\left|\frac{\partial \dot{q}_i}{\partial \dot{r}_k}\right| = \frac{\partial(\dot{q}_1, \cdots, \dot{q}_n)}{\partial(\dot{r}_1, \cdots, \dot{r}_n)} \equiv 0. \tag{22}$$

4. *Geometric Aspects of the Theory.* In this section we confine ourselves to the case $n=2$. The discussion will also apply to the case in which all but two of the q_i are not changed by the transformation. If

$$2T = a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2, \tag{23}$$

then T may be regarded as the kinetic energy of a particle of unit mass constrained to move in a surface whose metric is

$$ds^2 = a_{11}dq_1^2 + 2a_{12}dq_1dq_2 + a_{22}dq_2^2.$$

At any point P of the path π of the particle in this surface construct a geodesic G tangent to π. Let G cut any fixed directed curve Γ at N, and let O be a fixed point of Γ. Let $r_1 = ON$, measured along Γ, $r_2 = NP$, measured along G, $r_3 =$ angle between Γ and G. The equations of π, $q_i = q_i(t)$, are found by solving the given dynamical problem, while $\dot{q}_i$ are direction numbers of the tangent to π. The relations between q_i and r_3 are

$$q_i = q_i(r_1, r_2, r_3), \qquad (i = 1, 2). \tag{24}$$

If r_1 and r_2 are held fast, these are the equations of a geodesic circle, C, with center at N, and $\partial q_i/\partial r_3$ are direction numbers of the tangent to C. Now C is perpendicular to G and hence also to π. The latter fact is expressed by the equation

$$a_{ij}\dot{q}_i \frac{\partial q_j}{\partial r_3} \equiv 0. \tag{25}$$

By Theorem 2B this is a necessary and sufficient condition that

$$\left(\frac{\partial T'}{\partial \dot{r}_3}\right)_0 = 0. \tag{26}$$

Conversely if (26) holds, G will be tangent to π. These results may be stated as follows:

THEOREM 4. *If the kinetic energy of a dynamical system is given by*

$$2T = a_{ij}\dot{q}_i\dot{q}_j, \qquad (i, j = 1, 2),$$

and if on a surface whose metric is

$$ds^2 = a_{ij}dq_i dq_j, \qquad (i, j = 1, 2),$$

r_1, r_2, r_3 are three coordinates chosen in the manner specified above, then

$$\left(\frac{\partial T'}{\partial \dot{r}_3}\right)_0 \equiv 0.$$

Also, r_1, r_2, r_3 are obtained from the equations

$$\frac{d}{dt}\left(\frac{\partial L''}{\partial \dot{r}_s}\right) - \frac{\partial L''}{\partial r_s} = 0, \qquad (s = 1, 2, 3),$$

where the double prime denotes that the kinetic potential is expressed in terms of r_1, r_2, r_3, $\dot{r}_1$, $\dot{r}_2$.

5. *Examples.* For a unit particle constrained to move in the xy-plane, we have $2T = \dot{x}^2 + \dot{y}^2$. Let P be the point (x, y), N a point on the x-axis, and O the origin. Put $z = ON$, $\rho = NP$, $\phi = XNP$. Then

$$x = z + \rho\cos\phi, \qquad y = \rho\sin\phi.$$

If NP is to be tangent to the path of the particle, we must have

$$\dot{x}\frac{\partial x}{\partial \phi} + \dot{y}\frac{\partial y}{\partial \phi} = 0.$$

This is the condition expressed by (25). In terms of the new variables this condition becomes

$$\dot{z}\sin\phi - \rho\dot{\phi} = 0. \tag{27}$$

We easily find

$$2T' = \dot{z}^2 + 2\cos\phi\,\dot{z}\dot{\rho} + \dot{\rho}^2 - 2\rho\sin\phi\,\dot{z}\dot{\phi} + \rho^2\dot{\phi}^2,$$

and

$$\frac{\partial T'}{\partial \dot{\phi}} = -\rho\sin\phi\dot{z} + \rho^2\dot{\phi} = 0,$$

by (27). Using (27) we get

$$2T'' = (\dot{z}\cos\phi + \dot{\rho})^2. \tag{28}$$

It may be remarked that since (25) is linear and homogeneous in $\dot{q}_i$, T'' will always be a perfect square when $n=2$. If F is the force function expressed in terms of z, ρ, ϕ, the equations of motion are, by Theorem 4,

$$\frac{d}{dt}[\cos\phi(\dot{z}\cos\phi+\dot{\rho})] = \frac{\partial F}{\partial z},$$

$$\frac{d}{dt}(\dot{z}\cos\phi+\dot{\rho}) = \frac{\partial F}{\partial \rho},$$

$$\dot{z}\sin\phi(\dot{z}\cos\phi+\dot{\rho}) = \frac{\partial F}{\partial \phi}.$$

If we multiply these equations by ρ, $-\rho\cos\phi$, $\sin\phi$ respectively and add we obtain (27).

The total order of these equations appears to be five, but it is in reality only four since a first integral can be obtained from a linear combination of them without any integration. If the reader wishes to carry through the integration for the case $F=-gy=-g\rho\sin\phi$, he will see that only four arbitrary constants enter. The force function suggested is that due to gravity.

A second example is that given in the opening section. A careful comparison of this example with the theory is suggested.

6. *Case of more than One Extra Coordinate.* The preceding theory can be extended to the case in which more than one additional coordinate is introduced. If the equations of transformation are

$$q_i = q_i(r_1, \cdots, r_{n+p}), \qquad (i = 1, \cdots, n),$$

there are p extra variables. If the new variables are restricted by p equations like (14), and if as a consequence of these relations we have

$$\left(\frac{\partial T'}{\partial r_s}\right)_0 \equiv 0, \qquad (s = n+1, \cdots, n+p),$$

then it can be shown that

$$D_s L'' = 0, \qquad (s = 1, 2, \cdots, n+p). \tag{29}$$

No satisfactory geometrical interpretation has been found except for the case $p=n-1$. This includes also the case $p<n-1$, provided that $n-p-1$ of the original coordinates are not affected by the transformation. For example, in the case discussed in the first section $n=3$ and $p=1$ and there was one coordinate not affected by the transformation. The geometrical interpretation for $p=n-1$ is similar to that for $n=2$, $p=1$. If

$$2T = a_{ij}\dot{q}_i\dot{q}_j, \qquad (i, j = 1, 2, \cdots, n),$$

then T may be regarded as the kinetic energy of a unit particle moving in a space of n dimensions whose metric is

$$ds^2 = a_{ij}dq_idq_j, \qquad (i, j = 1, 2, \cdots, n).$$

Let $(r_1, \cdots, r_{n-1}, 0)$ represent the q-coordinates of a point N in this space; r_n = the distance along the geodesic joining N and the point P: $(q_1, \cdots, q_n)$; $r_{n+1}, \cdots, r_{2n-1}$ be the first $n-1$ direction cosines of this geodesic at N. Suppose the point N is so chosen that this geodesic will be tangent to the path at P. If then $r_1, \cdots, r_n$ are held fast in the equations of transformation,

$$q_i = q_i(r_1, \cdots, r_{2n-1}), \quad (i = 1, 2, \cdots, n),$$

these equations become those of a geodesic sphere. This sphere will be perpendicular to the geodesic NP at P, and hence also to the path. This fact is expressed by the $n-1$ equations

$$a_{ij}\dot{q}_i\frac{\partial q_j}{\partial r_s} = 0, \qquad (s = n+1, \cdots, 2n-1).$$

But these are necessary and sufficient that

$$\left(\frac{\partial T}{\partial \dot{r}_s}\right)_0 \equiv 0, \qquad (s = n+1, \cdots, 2n-1)$$

so that equations (29) will hold.

LEHIGH UNIVERSITY

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

The usual February Meeting in New York City will be omitted.

NEW YORK CITY, March 25-26, 1932.

Abstracts must be in the hands of Associate Secretary Tomlinson Fort, 501 West 116th St., New York City, not later than March 5. All such abstracts will appear in the March issue of the Bulletin. By invitation of the program committee, there will be a symposium on *Summability of series*, at which Professor Einar Hille will speak on *Summation of Fourier series*, Professor W. A. Hurwitz will speak on *General theorems on linear transformation of sequences*, and others will participate.

CHICAGO, April 8-9, 1932.

Abstracts must be in the hands of Associate Secretary Mark H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than March 18. All those received on or before March 6 will appear in the March issue of the Bulletin. By invitation of the program committee, Professor R. L. Wilder will deliver an address entitled *Point sets in three and higher dimensions and their investigation by means of unified analysis situs.*

On account of the fact that the Summer Meeting of the Society in 1932 will be held on the Pacific Coast, the meetings that were to have been held at Stanford University (April, 1932), at Seattle (June, 1932), and at Los Angeles (November, 1932) will be omitted.

LOS ANGELES, CALIFORNIA, SUMMER MEETING AND COLLOQUIUM, August 30-September 2, 1932.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wis., not later than July 26. Abstracts received by July 7 will appear in the July issue, or an earlier issue, of this Bulletin. Professor J. F. Ritt will deliver a series of colloquium lectures on *Systems of algebraic differential equations.*

NEW YORK CITY, October 29, 1932.

Abstracts are due by October 8, 1932.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, October, 1930, p. 58), are payable to the Treasurer of the Society, Professor G. W. Mullins, 501 West 116th St., New York City.

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